HARMONIC ANALYSIS

PIOTR HAJŁASZ

1. Maximal function

For a locally integrable function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal function* is defined by

$$\mathcal{M}f(x) = \sup_{r>0} ⨍_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

The operator $\mathcal{M}$ is not linear but it is subadditive. We say that an operator T from a space of measurable functions into a space of measurable functions is *subadditive* if

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)| \quad \text{a.e.}$$

and

$$|T(kf)(x)| = |k||Tf(x)| \quad \text{for } k \in \mathbb{C}.$$

The following integrability result, known also as the *maximal theorem*, plays a fundamental role in many areas of mathematical analysis.

Theorem 1.1 (Hardy-Littlewood-Wiener)**.** *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $\mathcal{M}f < \infty$ a.e. Moreover*

(a) *For $f \in L^1(\mathbb{R}^n)$*

$$|\{x : \mathcal{M}f(x) > t\}| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \quad \textit{for all } t > 0. \tag{1.1}$$

(b) *If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{M}f\|_p \leq 2 \cdot 5^{n/p} \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p \quad \textit{for } 1 < p < \infty,$$

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty \,.$$

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The estimate (1.1) is called *weak type estimate.*

Note that if $f \in L^1(\mathbb{R}^n)$ is a nonzero function, then $\mathcal{M}f \notin L^1(\mathbb{R}^n)$. Indeed, if $\lambda = \int_{B(0,R)} |f| > 0$, then for $|x| > R$ we have

$$\mathcal{M}f(x) \geq \fint_{B(x,R+|x|)} |f| \geq \frac{\lambda}{\omega_n(R+|x|)^n}\,,$$

and the function on the right hand side is not integrable on $\mathbb{R}^n$. Thus the statement (b) of the theorem is not true for $p = 1$.

If $g \in L^1(\mathbb{R}^n)$, then the *Chebyschev inequality*

$$|\{x:\,|g(x)| > t\}| \leq \frac{1}{t}\int_{\mathbb{R}^n} |g| \quad \text{for } t > 0$$

is easy to prove. Hence the inequality at (a) would follow from boundedness of of $\mathcal{M}f$ in L^1. Unfortunately $\mathcal{M}f$ is not integrable and (a) is the best what we can get for $p = 1$.

Before we prove the theorem we will show that it implies the Lebesgue differentiation theorem.

Theorem 1.2 (Lebesgue differentiation theorem)**.** *If* $f \in L^1_{\rm loc}(\mathbb{R}^n)$*, then*

$$\lim_{r\to 0}\fint_{B(x,r)} f(y)\,dy = f(x) \quad a.e.$$

Proof. Since the theorem is local in nature we can assume that $f \in L^1(\mathbb{R}^n)$. Let $f_r(x) = \fint_{B(x,r)} f(y)\,dy$ and define

$$\Omega f(x) = \limsup_{r\to 0} f_r(x) - \liminf_{r\to 0} f_r(x)\,.$$

It suffices to prove that $\Omega f = 0$ a.e. and that $f_r \to f$ in L^1. Indeed, the first property means that f_r converges a.e. to a measurable function g while the second one implies that for a subsequence $f_{r_i} \to f$ a.e. and hence $g = f$ a.e.

Observe that $\Omega f \leq 2\mathcal{M}f$ and hence for any $\varepsilon > 0$ Theorem 1.1(a) yields

$$|\{x:\,\Omega f(x) > \varepsilon\}| \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^n} |f|\,.$$

Let h be a continuous function such that $\|f-h\|_1 < \varepsilon^2$. Continuity of h implies $\Omega h = 0$ everywhere and hence

$$\Omega f \leq \Omega(f-h) + \Omega h = \Omega(f-h)\,,$$

so

$$|\{\Omega f > \varepsilon\}| \leq |\{\Omega(f-h) > \varepsilon\}| \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^n} |f-h| \leq C\varepsilon\,.$$

Since $\varepsilon > 0$ can be arbitrarily small we conclude $\Omega f = 0$ a.e. We are left with the proof that $f_r \to f$ in L^1. We have

$$\begin{aligned}\int_{\mathbb{R}^n} |f_r(x) - f(x)|\, dx &\leq \int_{\mathbb{R}^n} \fint_{B(x,r)} |f(y) - f(x)|\, dy\, dx \\ &= \int_{\mathbb{R}^n} \fint_{B(0,r)} |f(x+y) - f(x)|\, dy\, dx \\ &= \fint_{B(0,r)} \|f_y - f\|_1\, dy\,, \qquad (1.2)\end{aligned}$$

where $f_y(x) = f(x+y)$. Since $f_y \to f$ in L^1 as $y \to 0$ the right hand side of (1.2) converges to 0 as $r \to 0$. □

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we can define f at *every* point by the formula

$$f(x) := \limsup_{r\to 0} \fint_{B(x,r)} f(y)\, dy\,. \qquad (1.3)$$

According to the Lebesgue differentiation theorem this is a representative of f in the class of functions that coincide with f a.e.

DEFINITION. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that $x \in \mathbb{R}^n$ is a *Lebesgue point of f* if

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0\,,$$

where $f(x)$ is defined by (1.3).

Theorem 1.3. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the set of points that are not Lebesgue points of f has measure zero.*

Proof. For $c \in \mathbb{Q}$ let E_c be the set of points for which

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - c|\, dy = |f(x) - c| \qquad (1.4)$$

does not hold. Clearly $|E_c| = 0$ and hence the set $E = \bigcup_{c\in\mathbb{Q}} E_c$ has measure zero. Thus for $x \in \mathbb{R}^n \setminus E$ and all $c \in \mathbb{Q}$, (1.4) is satisfied. If $x \in \mathbb{R}^n \setminus E$ and $f(x) \in \mathbb{R}$, approximating $f(x)$ by rational numbers one can easily check that

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = |f(x) - f(x)| = 0\,.$$

The proof is complete. □

We can generalize the above result as follows. We say that $x \in \mathbb{R}^n$ is a *p-Lebesgue point* of $f \in L^p_{\mathrm{loc}}$, $1 \leq p < \infty$ if

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|^p\, dy = 0\,.$$

The same method as the one used above leads to the following result that we leave an an exercise.

Theorem 1.4. *If $f \in L^p_{\rm loc}(\mathbb{R}^n)$, $1 \le p < \infty$, then the set of points that are not p-Lebesgue points of f has measure zero.*

Definition. Let $E \subset \mathbb{R}^n$ be a measurable set. We say that $x \in \mathbb{R}^n$ is a density point of E if

$$\lim_{r\to 0} \frac{|B(x,r)\cap E|}{|B(x,r)|} = 1\,.$$

Applying the Lebesgue theorem to $f = \chi_E$ we obtain

Theorem 1.5. *Almost every point of a measurable set $E \subset \mathbb{R}^n$ is its density point and a.e. point of $\mathbb{R}^n \setminus E$ is not a density point of E.*

In the Lebesgue theorem we have seen that the averages of f over balls converge to $f(x)$ and it is natural to inquire if we can replace balls by other sets like cubes or even balls, but not centered at x.

Definition. We say that a family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^n$ is *regular at $x \in \mathbb{R}^n$* if

(a) The sets are bounded and have positive measure;
(b) There is a sequence $S_n \in \mathcal{F}$ with $|S_n| \to 0$;
(c) There is a constant $C > 0$ such that every $S \in \mathcal{F}$ is contained in a ball $B \supset S$ centered at x such that $|S| \ge C|B|$.

Example. The family of all cubes Q in $\mathbb{R}^n$ such that the distance of Q to x is no more than $C\mathrm{diam}\, Q$ is regular.

Theorem 1.6. *If $f \in L^1_{\rm loc}(\mathbb{R}^n)$, x is a Lebesgue point of f, and $\mathcal{F}$ is regular at x, then*

$$\lim_{\substack{S\in\mathcal{F}\\ |S|\to 0}} \fint_S f(y)\, dy = f(x)\,.$$

Proof. For $S \in \mathcal{F}$ denote by r_S the radius of a ball $B_S = B(x, r_S)$ such that $S \subset B_S$ and $|S| \ge C|B_S|$. Clearly if $|S| \to 0$, then $r_S \to 0$. We have

$$\begin{aligned} \left|\fint_S f(y)\, dy - f(x)\right| &\le \fint_S |f(y) - f(x)|\, dy \\ &\le |S|^{-1} \int_{B_S} |f(y) - f(x)|\, dy \\ &\le C^{-1} \fint_{B_S} |f(y) - f(x)|\, dy \to 0 \end{aligned}$$

as $|S| \to 0$. □

Note that if $\mathcal{F}$ is a regular family at 0 and we define the maximal function associated with $\mathcal{F}$ by

$$\mathcal{M}_{\mathcal{F}} f(x) = \sup_{S\in\mathcal{F}} \fint_S |f(x-y)|\, dy\,,$$

then it is a routine calculation to show that

$$\mathcal{M}_{\mathcal{F}} f(x) \leq C\mathcal{M} f(x)\,, \tag{1.5}$$

so $\mathcal{M}_{\mathcal{F}}$ satisfies the claim of Theorem 1.1 (with different constants). In particular $\mathcal{M}_{\mathcal{F}}$ is a bounded operator in L^p, $1 < p \leq \infty$.

In the proof of Theorem 1.1 we will need the following two results.

Theorem 1.7 (Cavalieri's principle)**.** *If μ is a σ-finite measure on X and $\Phi : [0,\infty) \to [0,\infty)$ is increasing, absolutely continuous and $\Phi(0) = 0$, then*

$$\int_X \Phi(|f|)\, d\mu = \int_0^\infty \Phi'(t)\mu(\{|f| > t\})\, dt\,.$$

Proof. The result follows immediately from the equality

$$\int_X \Phi(|f(x)|)\, d\mu(x) = \int_X \int_0^{|f(x)|} \Phi'(t)\, dt\, d\mu(x)$$

and the Fubini theorem. □

Corollary 1.8. *If μ is a σ-finite measure on X and $0 < p < \infty$, then*

$$\int_X |f|^p\, d\mu = p\int_0^\infty t^{p-1}\mu(\{|f| > t\})\, dt\,.$$

The next result has many applications that go beyond the maximal theorem.

Theorem 1.9 ($5r$-covering lemma)**.** *Let $\mathcal{B}$ be a family of balls in a metric space such that $\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$. Then there is a subfamily of pairwise disjoint balls $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B\,.$$

If the metric space is separable, then the family $\mathcal{B}'$ is countable and we can arrange it as a sequence $\mathcal{B}' = \{B_i\}_{i=1}^\infty$, so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^\infty 5B_i\,.$$

Remark. Here $\mathcal{B}$ can be either a family of open balls or closed balls. In both cases proof is the same.

Proof. Let $\sup\{\operatorname{diam} B:\ B\in\mathcal{B}\}=R<\infty$. Divide the family $\mathcal{B}$ according to the diameter of the balls

$$\mathcal{F}_j=\{B\in\mathcal{B}:\frac{R}{2^j}<\operatorname{diam} B\leq\frac{R}{2^{j-1}}\}\,.$$

Clearly $\mathcal{B}=\bigcup_{j=1}^\infty\mathcal{F}_j$. Define $\mathcal{B}_1\subset\mathcal{F}_1$ to be the maximal family of pairwise disjoint balls. Suppose the families $\mathcal{B}_1,\ldots,\mathcal{B}_{j-1}$ are already defined. Then we define $\mathcal{B}_j$ to be the maximal family of pairwise disjoint balls in

$$\mathcal{F}_j\cap\{B:\ B\cap B'=\emptyset\text{ for all }B'\in\bigcup_{i=1}^{j-1}\mathcal{B}_i\}\,.$$

Next we define $\mathcal{B}'=\bigcup_{j=1}^\infty\mathcal{B}_j$. Observe that every ball $B\in\mathcal{F}_j$ intersects with a ball in $\bigcup_{i=1}^j\mathcal{B}_j$. Suppose that $B\cap B_1\neq\emptyset$, $B_1\in\bigcup_{i=1}^j\mathcal{B}_i$. Then

$$\operatorname{diam} B\leq\frac{R}{2^{j-1}}=2\cdot\frac{R}{2^j}\leq 2\operatorname{diam} B_1$$

and hence $B\subset 5B_1$. □

Proof of Theorem 1.1. (a) Let $f\in L^1(\mathbb{R}^n)$ and $E_t=\{x:\mathcal{M}f(x)>t\}$. For $x\in E_t$, there is $r_x>0$ such that

$$⨍_{B(x,r_x)}|f|>t\,,$$

so

$$|B(x,r_x)|<t^{-1}\int_{B(x,r_x)}|f|\,.$$

Observe that $\sup_{x\in E_t}r_x<\infty$, because $f\in L^1(\mathbb{R}^n)$. The family of balls $\{B(x,r_x)\}_{x\in E_t}$ forms a covering of the set E_t, so applying the $5r$-covering lemma there is a sequence of pairwise disjoint balls $B(x_i,r_{x_i})$, $i=1,2,\ldots$ such that $E_t\subset\bigcup_{i=1}^\infty B(x_i,5r_{x_i})$ and hence

$$|E_t|\leq 5^n\sum_{i=1}^\infty|B(x_i,r_{x_i})|\leq\frac{5^n}{t}\sum_{i=1}^\infty\int_{B(x_i,r_{x_i})}|f|\leq\frac{5^n}{t}\int_{\mathbb{R}^n}|f|\,.$$

The proof is complete.

(b) Let $f\in L^p(\mathbb{R}^n)$. Since $\|\mathcal{M}f\|_\infty\leq\|f\|_\infty$ we can assume that $1<p<\infty$. Let $f=f_1+f_2$, where

$$f_1=f\chi_{\{|f|>t/2\}},\qquad f_2=f\chi_{\{|f|\leq t/2\}}$$

be a decomposition of f into its lower and upper parts. It is easy to check that $f_1\in L^1(\mathbb{R}^n)$. Since $|f|\leq|f_1|+t/2$ we have $\mathcal{M}f\leq\mathcal{M}f_1+t/2$ and hence

$$\{\mathcal{M}f>t\}\subset\{\mathcal{M}f_1>t/2\}\,.$$

Thus

$$
\begin{aligned}
(1.6) \qquad |E_t| &= |\{\mathcal{M}f > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\mathbb{R}^n} |f_1(x)|\, dx \\
&= \frac{2 \cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx\, .
\end{aligned}
$$

Cavalieri's principle gives

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\mathcal{M}f(x)|^p\, dx &= p \int_0^\infty t^{p-1} |\{\mathcal{M}f > t\}|\, dt \\
&\leq p \int_0^\infty t^{p-1} \left(\frac{2 \cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)\, dx \right) dt \\
&= 2 \cdot 5^n p \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2}\, dt\, dx \\
&= 2^p \cdot 5^n \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p\, dx
\end{aligned}
$$

and the results follows. □

Note that we proved in (1.6) the following inequality

$$
(1.7) \qquad |\{x : \mathcal{M}f(x) > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx
$$

which is slightly stronger than (1.1). We will need that inequality later.

For a positive measure μ on $\mathbb{R}^n$ we define the maximal function by

$$
\mathcal{M}\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{|B(x,r)|}\, .
$$

A minor modification of the proof of Theorem 1.1(a) leads to the following result.

Proposition 1.10. *If μ is a finite positive Borel measure on $\mathbb{R}^n$, then*

$$
|\{x : \mathcal{M}\mu(x) > t\}| \leq \frac{5^n}{t}\, \mu(\mathbb{R}^n) \quad \textit{for all } t > 0.
$$

Let $\mathcal{F}$ be the family of all rectangular boxes in $\mathbb{R}^n$ that contain the origin and have sides parallel to the coordinate axes. With the family we can associate the maximal function

$$
\widetilde{\mathcal{M}}f(x) = \sup_{S \in \mathcal{F}} ⨍_S |f(x-y)|\, dy\, .
$$

Note that the family $\mathcal{F}$ is *not* regular at 0 and hence the boundedness of $\mathcal{M}_{\mathcal{F}}$ in L^p, $1 < p \leq \infty$ cannot be concluded from (1.5). However, we have

Theorem 1.11 (Zygmund)**.** *For $1 < p < \infty$ there is a constant $C = C(n,p) > 0$ such that*

$$\|\widetilde{\mathcal{M}}f\|_p \leq C\|f\|_p\,. \tag{1.8}$$

Moreover, if $f \in L^p_{\rm loc}$, $1 < p < \infty$, then

$$\lim_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\,dy = f(x) \quad a.e. \tag{1.9}$$

Proof. First we will prove how to conclude (1.9) from (1.8). Note that since the family is not regular at 0, (1.9) is not a consequence of Theorem 1.6, see also Theorem 1.12. However

$$0 \leq \limsup_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\,dy - \liminf_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\,dy \leq 2\widetilde{\mathcal{M}}f(x)$$

and hence (1.9) follows from (1.8) by almost the same argument that was used to deduce the Lebesgue differentiation theorem from Theorem 1.1. We leave details to the reader. We are left with the proof of (1.8). For simplicity assume that $n = 2$. We have

$$\begin{aligned}
\widetilde{\mathcal{M}}f(x_1,x_2) &= \sup_{\substack{a_1,b_1>0\\ a_2,b_2>0}} \fint_{x_2-a_2}^{x_2+b_2}\fint_{x_1-a_1}^{x_1+b_1} |f(y_1,y_2)|\,dy_1\,dy_2 \\
&\leq \sup_{a_2,b_2>0} \fint_{x_2-a_2}^{x_2+b_2}\left(\sup_{a_1,b_1>0}\fint_{x_1-a_1}^{x_1+b_1} |f(y_1,y_2)|\,dy_1\right) dy_2\,.
\end{aligned}$$

On the right hand side we have iteration of one dimensional maximal functions. First we apply the maximal function to variable y_1 and evaluate it at x_1 and then we apply the maximal function to the variable y_2 and evaluate it at x_2. These are not exactly the Hardy-Littlewood maximal functions, because we take averages over all intervals that contain x_1 and then all the intervals that contain x_2, but these maximal functions are bounded by a constant multiplicity of the one dimensional Hardy-Littlewood maximal functions, see (1.5), because the family of all intervals that contain 0 is regular at 0. Thus it is easy to see that inequality (1.8) follows from the one dimensional version of Theorem 1.1 applied twice and the Fubini theorem. □

Surprisingly (1.9) does not hold for $p = 1$ and hence the maximal function $\widetilde{M}f$ does not satisfy the weak type estimate (1.1).[1] Namely one can prove the following result that we leave without a proof.

Theorem 1.12 (Saks)**.** *Let $\mathcal{F}$ be the family of all rectangular boxes in $\mathbb{R}^n$ that contain the origin and have sides parallel to the coordinate axes. Then*

[1]Such estimate would imply (1.9).

the set of functions $f \in L^1(\mathbb{R}^n)$ such that

$$\limsup_{\substack{\operatorname{diam} S \to 0 \\ S \in \mathcal{F}}} \fint_S f(x-y)\, dy = \infty \quad \text{for all } x \in \mathbb{R}^n$$

is a dense G_δ subset of $L^1(\mathbb{R}^n)$. In particular it is not empty.

The proof is based on the category method.

1.1. **The Calderón-Zygmund decomposition.** The following result plays a fundamental role in many areas of analysis.

Theorem 1.13 (Calderón-Zygmund decomposition)**.** *Suppose $f \in L^1(\mathbb{R}^n)$, $f \geq 0$ and $\alpha > 0$. Then there is an open set Ω and a closed set F such that*

(a) $\mathbb{R}^n = \Omega \cup F$, $\Omega \cap F = \emptyset$;
(b) $f \leq \alpha$ *a.e. in* F;
(c) Ω *can be decomposed into cubes* $\Omega = \bigcup_{k=1}^\infty Q_k$ *with pairwise disjoint interiors such that*

$$\alpha \leq \fint_{Q_k} f \leq 2^n \alpha\,, \quad k = 1, 2, 3, \ldots$$

Proof. Decompose $\mathbb{R}^n$ into a grid of identical cubes, large enough to have

$$\fint_Q f(x)\, dx \leq \alpha$$

for each cube in the grid. Take a cube Q from the grid and divide it into 2^n identical cubes. Let Q' be one of the cubes from this partition. We have two cases:

$$\fint_{Q'} f(x)\, dx > \alpha \qquad \text{or} \qquad \fint_{Q'} f(x)\, dx \leq \alpha\,.$$

If the fist case holds we include the open cube Q' to the family $\{Q_k\}$. Note that

$$\alpha < \fint_{Q'} f = 2^n |Q|^{-1} \int_{Q'} f \leq 2^n \fint_Q f \leq 2^n \alpha$$

so the condition (c) is satisfied. If the second case holds we divide Q' into 2^n identical cubes and proceed as above. We continue this process infinitely many times or until it is terminated. We apply it to all the cubes of the original grid. Let $\Omega = \bigcup_{k=1}^\infty Q_k$, where the cubes are defined by the first case of the process. It remains to prove that $f \leq \alpha$ a.e. in the set $\mathbb{R}^n \setminus \Omega$. The set F consists of faces of the cubes (this set has measure zero) and points x such that there is a sequence of cubes $\tilde{Q}_i$ with the property that $x \in \tilde{Q}_i$, $\operatorname{diam} \tilde{Q}_i \to 0$, $\fint_{\tilde{Q}_i} f \leq \alpha$. According to Theorem 1.6 for a.e. such x $\fint_{\tilde{Q}_i} f \to f(x)$ and hence $f \leq \alpha$ a.e. in F. □

Corollary 1.14. *Let f, α and Ω be as in Theorem 1.13. Then*

$$|\Omega| \leq \alpha^{-1}\|f\|_1 .$$

Proof. We have

$$|\Omega| = \sum_{k=1}^{\infty} |Q_k| \leq \sum_{k=1}^{\infty} \alpha^{-1} \int_{Q_k} |f| \leq \alpha^{-1}\|f\|_1 .$$

The proof is complete. □

We already observed that if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{M}f \notin L^1(\mathbb{R}^n)$, however we showed that the function cannot be globally integrable. It turns out that, in general, the maximal function need not be even locally integrable. We will actually characterize all functions such that the maximal function is locally integrable.

DEFINITION. We say that a measurable function f belongs to the *Zygmund space* $L \log L$ if $|f|\log(e + |f|) \in L^1$.

It is easy to see that for a space with finite measure we have

$$\bigcap_{p>1} L^p \subset L \log L \subset L^1 ,$$

so the Zygmund space is an intermediate space between all L^p for $p > 1$ and L^1.

Theorem 1.15 (Stein)**.** *Suppose that a measurable function f is supported in a ball B. Then $\mathcal{M}f \in L^1(B)$ if and only if $f \in L \log L(B)$.*

Proof. Suppose that $f \in L \log L(B)$. Then

$$\begin{aligned} \int_B \mathcal{M}f(x)\, dx &\leq |B| + \int_{\{\mathcal{M}f \geq 1\}} \mathcal{M}f(x)\, dx \\ &= |B| + \underbrace{|\{\mathcal{M}f \geq 1\}|}_{\leq |B|} + \int_1^{\infty} |\{\mathcal{M}f > t\}|\, dx . \end{aligned}$$

The last equality is a consequence of the Cavalieri principle. Applying inequality (1.7) we have

$$\begin{aligned} \int_B \mathcal{M}f(x)\, dx &\leq 2|B| + \int_1^{\infty} \left(\frac{C}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx \right) dt \\ &= 2|B| + \int_B \left(\int_1^{\max\{2|f(x)|,1\}} \frac{dt}{t} \right) |f(x)|\, dx \\ &\leq 2|B| + \int_B |f(x)| \log(e + 2|f(x)|)\, dx < \infty . \end{aligned}$$

This proves the implication from right to left.

To prove the implication from left to right we will show first an inequality which is, in some sense, an inverse inequality to (1.7). Namely we will prove that there is a constant $C = C(n) > 0$ such that

$$|\{x : \mathcal{M}f(x) > Ct\}| \geq \frac{2^{-n}}{t} \int_{\{|f|>t\}} |f(x)|\, dx\,. \tag{1.10}$$

To this end we need to apply the Calderón-Zygmund decomposition to the function $|f|$ and $\alpha = t$. If $\Omega = \bigcup_k Q_k$ is the Calderón-Zygmund decomposition, then

$$t < \fint_{Q_k} |f| \leq 2^n t$$

and hence $\mathcal{M}f(x) > Ct$ for all $x \in Q_k$. Thus

$$|\{x : \mathcal{M}f(x) > Ct\}| \geq \sum_{k=1}^{\infty} |Q_k| \geq \frac{2^{-n}}{t} \int_{\Omega} |f(x)|\, dx\,.$$

Since $|f| \leq t$ for $x \notin \Omega$ we have

$$\int_{\Omega} |f(x)|\, dx \geq \int_{\{|f|>t\}} |f(x)|\, dx\,.$$

The last two inequalities combined together prove (1.10). Note that the inequality (1.10) is satisfied by an arbitrary function $f \in L^1(\mathbb{R}^n)$. Suppose now that f vanished outside a ball B and that $\mathcal{M}f \in L^1(B)$. Observe that it is not clear if in the inequality (1.10) we can replace the set $\{x : \mathbb{R}^n : \mathcal{M}f(x) > Ct\}$ on the left hand side by the set $\{x \in B : \mathcal{M}f(x) > Ct\}|$. Indeed, the maximal function does not vanish outside B and the proof involved estimates on $\mathbb{R}^n$.

Note that $\mathcal{M}f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, it is integrable in B and locally bounded outside the closed ball B, so we need to verify integrability of $\mathcal{M}f$ in a neighborhood of the boundary of B, but it is easy to see that if x is near the boundary and outside the ball, we can estimate the value of $\mathcal{M}f(x)$ by (constant times) the value of the maximal function in a point being the reflection of x across the boundary. Thus the integrability of $\mathcal{M}f$ near the boundary follows from the integrability of $\mathcal{M}f$ in B. Note also that the set $\{\mathcal{M}f > 1\}$ is bounded, because $\mathcal{M}f(x)$ decays to zero as $x \to \infty$. Thus local integrability of $\mathcal{M}f$ and boundedness of the set $\{\mathcal{M}f > 1\}$ implies that the function $\mathcal{M}f$ in integrable in $\{\mathcal{M}f > 1\}$. With these remarks we can complete the proof as follows.

$$\begin{aligned}\infty &> \int_{\{\mathcal{M}f>1\}} \mathcal{M}f(x)\,dx \\ &\geq \int_1^\infty |\{\mathcal{M}f>t\}|\,dt \\ &\geq \int_1^\infty \frac{C2^{-n}}{t}\int_{\{|f|>t/C\}}|f(x)|\,dx\,dt \\ &\geq C2^{-n}\int_B\left(\int_1^{\max\{C|f(x)|,1\}}\frac{dt}{t}\right)|f(x)|\,dx \\ &= C2^{-n}\int_B |f(x)|\log(\max\{C|f(x)|,1\})\,dx\,.\end{aligned}$$

The proof is complete. □

A more careful analysis leads to the following version of Stein's theorem. In an open set $\Omega\subset\mathbb{R}^n$ we define the local maximal function by

$$\mathcal{M}_\Omega f = \sup\{ ⨍_Q |f| : x\in Q\subset\Omega\}\,,$$

where the supremum is taken over all cubes Q in Ω that contain x.

Theorem 1.16 (Stein)**.** *Let $Q\subset\mathbb{R}^n$ be a cube. Then $\mathcal{M}_Q f\in L^1(Q)$ if and only if $f\in L\log L(Q)$. Moreover*

$$5^{-(n+1)}\int_Q \mathcal{M}_Q f \leq \int_Q |f|\log\left(e+\frac{|f|}{|f|_Q}\right) \leq 2^{n+2}\int_Q \mathcal{M}_Q f\,,$$

where

$$|f|_Q = ⨍_Q |f|\,.$$

1.2. **Fractional integration theorem.** As an application of the Hardy-Littlewood-Wiener theorem we will prove a result due to Hardy-Littlewood-Sobolev about integrability of Riesz potentials, called *fractional integration theorem.*

For $0<\alpha<n$ and $n\geq 2$ we define the *Riesz potentials* by

$$(I_\alpha f)(x) = \frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy\,,$$

where

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}}\,2^\alpha\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\,.$$

At this moment the particular value of the constant $\gamma(\alpha)$ is not important to us. We could even replace this constant by 1.

Theorem 1.17 (Hardy-Littlewood-Sobolev)**.** *Let $\alpha > 0$, $1 < p < \infty$ and $\alpha p < n$. Then there is a constant $C = C(n, p, \alpha)$ such that*

$$\|I_\alpha f\|_{p^*} \leq C\|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n)\,,$$

where $p^* = np/(n - \alpha p)$.

We precede the proof with a technical lemma.

Lemma 1.18. *If $0 < \alpha < n$, and $\delta > 0$, then there is a constant $C = C(n, \alpha)$ such that*

$$\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \leq C\delta^\alpha \mathcal{M}f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. For $x \in \mathbb{R}^n$ and $\delta > 0$ consider the annuli

$$A(k) = B\left(x, \frac{\delta}{2^k}\right) - B\left(x, \frac{\delta}{2^{k+1}}\right)\,.$$

We have

$$\begin{aligned}
\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy &= \sum_{k=0}^\infty \int_{A(k)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\
&\leq \sum_{k=0}^\infty \left(\frac{\delta}{2^{k+1}}\right)^{\alpha-n} \int_{A(k)} |f(y)|\, dy \\
&\leq \omega_n \sum_{k=0}^\infty \left(\frac{\delta}{2^{k+1}}\right)^{\alpha-n} \left(\frac{\delta}{2^k}\right)^n ⨍_{B(x,\delta/2^k)} |f(y)|\, dy \\
&\leq \omega_n \delta^\alpha \left(\frac{1}{2}\right)^{\alpha-n} \left(\sum_{k=0}^\infty \frac{1}{2^{k\alpha}}\right) \mathcal{M}f(x)\,.
\end{aligned}$$

The proof is complete. ∎

Proof of Theorem 1.17. Fix $\delta > 0$. Hölder's inequality and integration in polar coordinates yield

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy &\leq \|f\|_p \left(\int_{\mathbb{R}^n \setminus B(x,\delta)} \frac{dy}{|x-y|^{(n-\alpha)p'}}\right)^{1/p'} \\
&= \|f\|_p \left(n\omega_n \int_\delta^\infty s^{n-1-(n-\alpha)p'}\, ds\right)^{1/p'} \\
&= C(n,p,\alpha)\delta^{\alpha-(n/p)}\|f\|_p\,,
\end{aligned}$$

because $n\omega_n$ equals the $(n-1)$-dimensional measure of the unit sphere S^{n-1} and $n - (n-\alpha)p' < 0$. Thus the lemma gives

$$|I_\alpha f(x)| \leq C\left(\delta^\alpha M f(x) + \delta^{\alpha-(n/p)}\|f\|_p\right)\,.$$

Taking

$$\delta = \left(\frac{\mathcal{M}f(x)}{\|f\|_p}\right)^{-p/n}$$

we obtain

$$|I_\alpha f(x)| \leq C(\mathcal{M}f(x))^{1-\frac{\alpha p}{n}} \|f\|_p^{\frac{\alpha p}{n}}$$

which is equivalent to

$$|I_\alpha f(x)|^{p^*} \leq C(\mathcal{M}f(x)|^p \|f\|_p^{\frac{\alpha p}{n} p^*}\,.$$

Integrating both sides over $\mathbb{R}^n$ and applying boundedness of the maximal function in L^p yields the result. □

2. Fourier transform

2.1. Measures and convolution.

Theorem 2.1 (Minkowski's integral inequality)**.** *If μ and ν are σ-finite measures on X and Y respectively and if $F : X \times Y \to \mathbb{R}$ is measurable, then for $1 \leq p < \infty$ we have*

$$\left(\int_Y \left(\int_X |F(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \leq \int_X \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x)\,.$$

Proof.

$$\begin{aligned}
&\left(\int_Y \left(\int_X |F(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \\
&= \sup_{\substack{h\in L^q(\nu)\\ \|h\|_q=1}} \int_Y h(y) \left(\int_X |F(x,y)|\, d\mu(x)\right) d\nu(y) \\
&= \sup_{\substack{h\in L^q(\nu)\\ \|h\|_q=1}} \int_X \left(\int_Y h(y)|F(x,y)|\, d\nu(y)\right) d\mu(x) \\
&\leq \sup_{\substack{h\in L^q(\nu)\\ \|h\|_q=1}} \int_X \underbrace{\left(\int_Y |h(y)|^q\, d\nu(y)\right)^{1/q}}_{1} \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x) \\
&= \int_X \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x)\,.
\end{aligned}$$

The proof is complete. □

Exercise. *Show that the classical Minkowski inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ follows from the integral Minkowski inequality.*

Recall that the convolution of measurable functions on $\mathbb{R}^n$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy\, .$$

Theorem 2.2. *If $1 \le p < \infty$, then*

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

Proof. The inequality can be obtained from the Minkowski integral inequality as follows

$$\begin{aligned}
\|f * g\|_p &= \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)g(y)\, dy \Big|^p dx \Big)^{1/p} \\
&\le \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)\, \underbrace{|g(y)|\, dy}_{d\mu} \Big)^p \underbrace{dx}_{d\nu} \Big)^{1/p} \\
&\le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)|^p\, dx \Big)^{1/p} |g(y)|\, dy \\
&= \|f\|_p \|g\|_1\, .
\end{aligned}$$

The proof is complete. □

The above result is a special case of a more general inequality.

Theorem 2.3 (Young's inequality)**.** *If $1 \le p, q, r \le \infty$ and $q^{-1} = p^{-1} + r^{-1} - 1$, then*

$$\|f * g\|_q \le \|f\|_p \|g\|_r\, .$$

Exercise. *Prove it.*

Recall that $C_0(\mathbb{R}^n)$ is the space of continuous functions vanishing at infinity, i.e. $f \in C_0(\mathbb{R}^n)$ if f is continuous and

$$\lim_{|x|\to\infty} f(x) = 0.$$

$C_0(\mathbb{R}^n)$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$ and $C_0^\infty(\mathbb{R}^n)$ (compactly supported smooth functions) is a dense subset of $C_0(\mathbb{R}^n)$.

If μ is a signed (Borel) measure on $\mathbb{R}^n$, then there is a unique Hahn decomposition

$$\mu = \mu^+ - \mu^-,$$

where μ^+ and μ^- are positive Borel measures concentrated on disjoint sets. We define the measure $|\mu|$ as

$$|\mu| = \mu^+ + \mu^-.$$

The number $\|\mu\| = |\mu|(\mathbb{R}^n)$ us called *total variation* of μ. The measures of finite total variation form Banach space with the norm $\|\mu\|$. We denote it by $\mathcal{B}(\mathbb{R}^n)$.

If $f\in L^1(\mathbb{R}^n)$, then $d\mu=f(x)\,dx$ is a measure of finite total variation $\mu(E)=\int_E f(x)\,dx$, $|\mu|(E)=\int_E|f|\,dx$, $\|\mu\|=\|f\|_1$. Thus $L^1(\mathbb{R}^n)$ can be identified as a closed subspace of $\mathcal{B}(\mathbb{R}^n)$ by the isometry

$$L^1(\mathbb{R}^n)\ni f\mapsto f(x)\,dx\in\mathcal{B}(\mathbb{R}^n).$$

Theorem 2.4 (Riesz representation theorem)**.** *The dual space to $C_0(\mathbb{R}^n)$ is isometrically isomorphic to the space of measures of finite total variation. More precisely, if $\Phi\in(C_0(\mathbb{R}^n))^*$, then there is a unique measure μ of finite total variation such that*

$$\Phi(f)=\int_{\mathbb{R}^n}f\,d\mu\quad\textit{for } f\in C_0(\mathbb{R}^n).$$

Moreover $\|\Phi\|=\|\mu\|=|\mu|(\mathbb{R}^n)$.

If $f,g\in L^1(\mathbb{R}^n)$, then $f*g\in L^1(\mathbb{R}^n)\subset\mathcal{B}(\mathbb{R}^n)$ and hence it acts as a functional on $C_0(\mathbb{R}^n)$ by the formula

$$\begin{aligned}\Phi(h)&=\int_{\mathbb{R}^n}h(x)(f*g)(x)\,dx=\int_{\mathbb{R}^n}h(x)\Big(\int_{\mathbb{R}^n}f(x-y)g(y)\,dy\Big)\,dx\\&=\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}h(x)f(x-y)\,dx\Big)\,g(y)\,dy\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}h(x+y)f(x)g(y)\,dx\,dy\,.\end{aligned}$$

This suggests how to define convolution of measures.

If $\mu_1,\mu_2\in\mathcal{B}(\mathbb{R}^n)$, then

$$\Phi(h)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}h(x+y)\,d\mu_1(x)\,d\mu_2(y)$$

defines a functional on $C_0(\mathbb{R}^n)$ and hence there is a unique measure $\mu\in\mathcal{B}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}h(x+y)\,d\mu_1(x)\,d\mu_2(y)=\int_{\mathbb{R}^n}h(x)\,d\mu(x)\quad\text{for all } h\in C_0(\mathbb{R}^n).$$

We denote

$$\mu=\mu_1*\mu_2$$

and call it *convolution of measures.* Clearly

$$\mu_1*\mu_2=\mu_2*\mu_1\quad\text{and}\quad\|\mu_1*\mu_2\|\leq\|\mu_1\|\,\|\mu_2\|.$$

If $d\mu_1=f\,dx$, $d\mu_2=g\,dx$, then

$$\mu_1*\mu_2=(f*g)\,dx,$$

so the convolution of measures extends the notion of convolution of functions. If $d\mu_1 = f\,dx$ and $\mu \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)\,d\mu_1(x)\,d\mu(y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)f(x)\,dx\,d\mu(y)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)\,dx\,d\mu(y)\\ &= \int_{\mathbb{R}^n} h(x)\Big(\int_{\mathbb{R}^n} f(x-y)\,d\mu(y)\Big)\,dx.\end{aligned}$$

Thus $\mu_1 * \mu$ can be identified with a function

$$x \mapsto \int_{\mathbb{R}^n} f(x-y)\,d\mu(y) \in L^1(\mathbb{R}^n),$$

so we can write

$$f * \mu = \int_{\mathbb{R}^n} f(x-y)\,d\mu(y), \quad \|f*\mu\|_1 \le \|f\|_1\,\|\mu\|.$$

Theorem 2.5. *If $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\mu \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|f*\mu\|_p \le \|f\|_p\,\|\mu\|.$$

Proof is almost the same as that for Theorem 2.2 as we leave it to the reader.

Exercise. *Find $\delta_a * \delta_b$.*

2.2. **Fourier transform.** For $f \in L^1(\mathbb{R}^n)$ we define the *Fourier transform* as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$$

where

$$x\cdot\xi = \sum_{j=1}^n x_j\xi_j\,.$$

If $\mu \in \mathcal{B}(\mathbb{R}^n)$ then we define

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\,d\mu(x).$$

For $f \in L^p(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ we define

$$\tau_h f(x) = f(x+h).$$

We will frequently use the following well known result.

Lemma 2.6. *For $1 \le p < \infty$*

$$\|f - \tau_h f\|_p \to 0 \quad \textit{as } |h| \to 0.$$

Theorem 2.7. *The Fourier transform has the following properties*

(a)
$$\wedge : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$$
is a bounded linear operator with
$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

(b)
$$(f * g)\hat{} = \hat{f}\hat{g} \quad \textit{for } f, g \in L^1(\mathbb{R}^n).$$

(c) *If* $f, g \in L^1(\mathbb{R}^n)$, *then* $\hat{f}g, f\hat{g} \in L^1(\mathbb{R}^n)$ *and*
$$\int_{\mathbb{R}^n} \hat{f}(x)g(x)\, dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\, dx$$

(d)
$$(\tau_h f)\hat{}(\xi) = \hat{f}(\xi)e^{2\pi i h\cdot\xi}$$
$$(f(x)e^{2\pi i h\cdot x})\hat{}(\xi) = \hat{f}(\xi - h).$$

(e) *If* $f_\varepsilon(x) = \varepsilon^{-n} f(x/\varepsilon)$, *then*
$$(f_\varepsilon)\hat{}(\xi) = \hat{f}(\varepsilon\xi), \quad (f(\varepsilon x))\hat{}(\xi) = (\hat{f})_\varepsilon(\xi)\,.$$

(f) *If* $\rho \in O(n)$ *is an orthogonal transformation, then*
$$(f(\rho\,\cdot))\hat{}(\xi) = \hat{f}(\rho\xi).$$

Proof. (a) Clearly $\wedge : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is a bounded linear mapping with $\|\hat{f}\|_\infty \leq \|f\|_1$. Indeed,
$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} \left|f(x)e^{-2\pi i x\cdot\xi}\right| dx = \|f\|_1.$$
The dominate convergence theorem implies that the function $\hat{f}$ is continuous. It remains to prove that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. Let $\xi \neq 0$. Since $e^{\pi i} = -1$ we have
$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx = -\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}e^{\pi i}\, dx \\
&= -\int_{\mathbb{R}^n} f(x)\exp\Big(-2\pi i\Big(x - \frac{\xi}{2|\xi|^2}\Big)\cdot\xi\Big)\, dx \\
&= -\int_{\mathbb{R}^n} f\Big(x + \frac{\xi}{2|\xi|^2}\Big)\, e^{-2\pi i x\cdot\xi}\, dx\,.
\end{aligned}$$
Hence
$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(f(x) - f\Big(x + \frac{\xi}{2|\xi|^2}\Big)\Big)\, e^{-2\pi i x\cdot\xi}\, dx$$
and thus
$$|\hat{f}(\xi)| \leq \frac{1}{2}\left\|f - \tau_{\frac{\xi}{2|\xi|^2}} f\right\|_1 \to 0 \quad \text{as } |\xi| \to \infty.$$

(b)

$$\begin{aligned}(f*g)\hat{}(\xi) &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\Big)\,e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)e^{-2\pi i(x-y)\cdot\xi}\,dx\Big)\,g(y)e^{-2\pi i y\cdot\xi}\,dy\\ &= \hat{f}(\xi)\hat{g}(\xi)\,.\end{aligned}$$

(c) $\hat{f}g, f\hat{g}\in L^1$, because the functions $\hat{f},\hat{g}$ are bounded and the equality of the integrals easily follows from the Fubini theorem.

(d)

$$\begin{aligned}(\tau_h f)\hat{}(\xi) &= \int_{\mathbb{R}^n} f(x+h)e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i(x-h)\cdot\xi}\,dx\\ &= e^{2\pi i h\cdot\xi}\underbrace{\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx}_{\hat{f}(\xi)}\,.\end{aligned}$$

The second equality follows from a similar argument.

(e)

$$\begin{aligned}(f_\varepsilon)\hat{}(\xi) &= \int_{\mathbb{R}^n}\varepsilon^{-n}f\Big(\frac{x}{\varepsilon}\Big)\,e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i(\varepsilon y)\cdot\xi}\,dy\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot(\varepsilon\xi)}\,dy\\ &= \hat{f}(\varepsilon\xi)\,.\end{aligned}$$

The second formula follows from the first one if we replace ε by ε^{-1}.

(f)

$$\begin{aligned}(f(\rho\,\cdot))\hat{}(\xi) &= \int_{\mathbb{R}^n} f(\rho x)e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i(\rho^{-1}y)\cdot\xi}\,dy\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot(\rho\xi)}\,dy\\ &= \hat{f}(\rho\xi)\,.\end{aligned}$$

Equality $(\rho^{-1}y)\cdot\xi = y\cdot(\rho\xi)$ follows from the fact that the mapping $x\mapsto\rho x$ is an isometry. □

Theorem 2.8. *Suppose $f \in L^1(\mathbb{R}^n)$ and $x_k f(x) \in L^1(\mathbb{R}^n)$, where x_k is the k-th coordinate function. Then $\hat{f}$ is differentiable with respect to ξ_k and*

$$(-2\pi i x_k f(x))\hat{} = \frac{\partial \hat{f}}{\partial \xi_k}(\xi)\,.$$

Proof. Let e_k be the unit vector along the k-th coordinate. Then the second part of Theorem 2.7(d) gives

$$\frac{\hat{f}(\xi + he_k) - \hat{f}(\xi)}{h} = \left(\frac{e^{-2\pi i(he_k)\cdot x} - 1}{h}\, f(x)\right)^\wedge (\xi) \to (-2\pi i x_k f(x))\hat{}(\xi)\,.$$

The convergence follows from the continuity of the Fourier transform in L^1. □

Definition. We say that f is *differentiable in the L^p norm with respect to x_k* if $f \in L^p(\mathbb{R}^n)$ and there is $g \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} - g(x) \right|^p dx \to 0 \quad \text{as } h \to 0.$$

The function g is called the *partial derivative of f (with respect to x_k) in the L^p norm.* We denote it by $g = \partial f/\partial x_k$.

Theorem 2.9. *If $f \in L^1(\mathbb{R}^n)$ and $\partial f/\partial x_k$ is the partial derivative of f in the L^1 norm, then*

$$\left(\frac{\partial f}{\partial x_k}\right)^\wedge = 2\pi i \xi_k \hat{f}(\xi)\,.$$

Proof. The first part of Theorem 2.7(d) gives

$$\left(\frac{\partial f}{\partial x_k}\right)^\wedge - \hat{f}(\xi)\,\frac{e^{2\pi i(he_k)\cdot\xi} - 1}{h} = \left(\frac{\partial f}{\partial x_k} - \frac{f(x+he_k) - f(x)}{h}\right)^\wedge \to 0$$

as $h \to 0$, so

$$\left(\frac{\partial f}{\partial x_k}\right)^\wedge (\xi) = \lim_{h\to 0} \hat{f}(\xi)\,\frac{e^{2\pi i(he_k)\cdot\xi} - 1}{h} = 2\pi i \xi_k \hat{f}(\xi)\,.$$

The proof is complete. □

With each polynomial

$$P(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha$$

of variables $x_1, \ldots, x_n$ we associate a differential operator

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha = \sum_{|\alpha|\le m} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\,.$$

Then under suitable assumptions Theorems 2.8 and 2.9 have the following higher order generalizations

$$P(D)\hat{f}(\xi) = (P(-2\pi ix)f(x))\hat{}(\xi), \quad (P(D)f)\hat{}(\xi) = P(2\pi i\xi)\hat{f}(\xi). \tag{2.1}$$

2.3. **Summability methods.** An important problem is a search for the inversion formula. Namely, given the Fourier transform $\hat{f}$, how can we find a formula for f? We would like to prove that

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi ix\cdot\xi}\, d\xi\,.$$

As we will see later the formula is true under very restrictive assumptions, but not always. Indeed, in general the right hand side makes no sense as the Fourier transform of an integrable function is not necessarily integrable. To handle this problem we have to use so called summability methods.

The two most important summability methods are the Abel and the Gauss (Gauss-Weierstrass) methods.

For each $\varepsilon > 0$ the *Abel mean* of a function f is

$$A_\varepsilon(f) = \int_{\mathbb{R}^n} f(x)e^{-\varepsilon|x|}\, dx\,.$$

If f is integrable, then

$$\lim_{\varepsilon\to 0} A_\varepsilon(f) = \int_{\mathbb{R}^n} f(x)\, dx\,.$$

However the integral $A_\varepsilon(f)$ exists also for non integrable functions. For example it exists if f is bounded. If the limit $\lim_{\varepsilon\to 0} A_\varepsilon(f) = \ell$ exists and is finite we say that $\int_{\mathbb{R}^n} f$ is *Abel summable to* ℓ.

Exercise. *Prove that if* $\lim_{a\to\infty}\int_0^a f(x)\, dx = \ell$, *then* $A_\varepsilon = \int_0^\infty f(x)e^{-\varepsilon x}\, dx$ *converges to* ℓ.

The *Gauss mean* of f is

$$G_\varepsilon(f) = \int_{\mathbb{R}^n} f(x)\, e^{-\varepsilon|x|^2}\, dx\,.$$

We say that $\int_{\mathbb{R}^n} f(x)\, dx$ is *Gauss summable* to ℓ if $\lim_{\varepsilon\to 0} G_\varepsilon(f) = \ell$.

The two methods can be put is a more general framework. If $\Phi \in C_0(\mathbb{R}^n)$ and $\Phi(0) = 1$, then we define the Φ-*mean* by

$$M_{\varepsilon,\Phi}(f) = M_\Phi(f) = \int_{\mathbb{R}^n} f(x)\Phi(\varepsilon x)\, dx\,.$$

If $\lim_{\varepsilon\to 0} M_\varepsilon(f) = \ell$, then we say that $\int_{\mathbb{R}^n} f$ is Φ-*summable to* ℓ.

Our aim is to apply the summability methods to reconstruct f from $\hat{f}$. More precisely we want to investigate functions Φ such that Φ-means

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \Phi(\varepsilon\xi)\, d\xi$$

converge in a certain sense to $f(x)$.

Let $f, \Phi \in L^1(\mathbb{R}^n)$ and $\varphi = \hat{\Phi}$. Fix $t \in \mathbb{R}^n$ and define

$$g(x) = e^{2\pi i x\cdot t} \Phi(\varepsilon x) \in L^1(\mathbb{R}^n)\,.$$

Then Theorem 2.7(d,e) gives

$$\hat{g}(\xi) = (\hat{\Phi})_\varepsilon(\xi - t) = \varphi_\varepsilon(\xi - t).$$

Hence Theorem 2.7(c) implies

$$\int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i x\cdot t} \Phi(\varepsilon x)\, dx = \int_{\mathbb{R}^n} f(x) \varphi_\varepsilon(x - t)\, dx\,.$$

Replacing x by ξ in the left integral and x by y in the right integral and finally replacing t by x we have

Theorem 2.10. *If $f, \Phi \in L^1(\mathbb{R}^n)$ and $\varphi = \hat{\Phi}$, then*

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \Phi(\varepsilon\xi)\, d\xi = \int_{\mathbb{R}^n} f(y) \varphi_\varepsilon(y - x)\, dy\,.$$

To see that the right hand side converges in a certain sense to $f(x)$ and $\varepsilon \to 0$ for a reasonable class of functions φ we need the next result.

Theorem 2.11. *Suppose $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ or $f \in C_0(\mathbb{R}^n)$, $p = \infty$, then*

$$\|f * \varphi_\varepsilon - f\|_p \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. By a change of variables $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$. Hence

$$(f * \varphi_\varepsilon)(x) - f(x) = \int_{\mathbb{R}^n} (f(x - y) - f(x)) \varphi_\varepsilon(y)\, dy\,.$$

Therefore the Minkowski integral inequality yields

$$\begin{aligned} \|f * \varphi_\varepsilon - f\|_p &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x - y) - f(x)) \varphi_\varepsilon(y)\, dy \right|^p dx \right)^{1/p} \\ &\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x - y) - f(x)|^p\, dx \right)^{1/p} \varepsilon^{-n} |\varphi(y/\varepsilon)|\, dy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x - \varepsilon y) - f(x)|^p\, dx \right)^{1/p} |\varphi(y)|\, dy \\ &= \int_{\mathbb{R}^n} \omega(-\varepsilon y) |\varphi(y)|\, dy \qquad (2.2) \end{aligned}$$

where

$$\omega(h) = \Big(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx \Big)^{1/p}.$$

Clearly $\omega(h) \le 2\|f\|_p$ and $\omega(h) \to 0$ as $h \to 0$ by Lemma 2.6, so the right hand side of (2.2) converges to 0 by the Lebesgue dominated convergence theorem. The case $f \in C_0(\mathbb{R}^n)$ with $p = \infty$ follows from a similar argument. □

Corollary 2.12. *Suppose $\varphi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 0$. Then $\|f * \varphi_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$ whenever $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ or $f \in C_0(\mathbb{R}^n)$, $p = \infty$.*

Proof. Note that

$$(f * \varphi_\varepsilon)(x) = (f * \varphi_\varepsilon)(x) - f(x) \cdot 0 = \int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)\, dy$$

and the rest follows by the same argument as in the proof of Theorem 2.11. □

Now we can prove a general result about the inversion formula in terms of the summability methods.

Theorem 2.13. *If $\Phi \in L^1(\mathbb{R}^n)$ and $\varphi = \hat{\Phi} \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$, then the Φ-means of the integral $\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$ converge to $f(x)$ in the L^1 norm, i.e. if*

$$M_\varepsilon(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, \Phi(\varepsilon\xi)\, d\xi$$

then

$$M_\varepsilon \to f \quad \text{in } L^1(\mathbb{R}^n).$$

Proof. It is a direct consequence of Theorem 2.10 and Theorem 2.11. □

The existence of functions Φ satisfying the assumptions of Theorem 2.13 follows from the next result. Note that the result also shows a function which is a fixed point of the Fourier transform.

Theorem 2.14. *Let $f(x) = e^{-4\pi^2 t|x|^2}$, $t > 0$. Then*

(a) $W(x,t) := \hat{f}(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$.
(b) *The function W has the following scaling property with respect to t: if $\varphi(x) = W(x,1)$, then $W(x,t) = \varphi_{t^{1/2}}(x)$.*
(c)
$$\int_{\mathbb{R}^n} W(x,t)\, dx = 1 \quad \text{for all } t > 0.$$

In particular, if $f(x) = e^{-\pi|x|^2}$, then $\hat{f}(x) = e^{-\pi|x|^2}$, i.e. the function f is a fixed point of the Fourier transform.

Proof. (a) By a simple change of variables it suffices to prove the formula for the Fourier transform for $t = (4\pi)^{-1}$. If $u(x) = e^{-\pi x^2}$ is a function of one variable, then

$$u' = -2\pi x u, \quad u' = -i(-2\pi i x u)$$

and hence

$$(u')\hat{} = -i(-2\pi i x u(x))\hat{}(\xi).$$

Applying Theorems 2.8 and 2.9 yields

$$2\pi i\xi \hat{u}(\xi) = -i(\hat{u})'(\xi)$$

$$(\hat{u})'(\xi) = -2\pi\xi\hat{u}(\xi).$$

Solving this differential equation gives

$$\hat{u}(\xi) = \hat{u}(0)e^{-\pi\xi^2}.$$

Since

$$\hat{u}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$$

we obtain $\hat{u}(\xi) = e^{-\pi\xi^2}$. If $f(x) = e^{-\pi|x|^2}$ is a function of several variables, then

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi}\, dx = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-\pi x_k^2} e^{-2\pi i x_k \xi_k}\, dx_k \\ &= \prod_{k=1}^n \hat{u}(\xi_k) = \prod_{k=1}^n e^{-\pi\xi_k^2} = e^{-\pi|\xi|^2}. \end{aligned}$$

(b) is obvious.

(c) It follows from the scaling property (b) that

$$\int_{\mathbb{R}^n} W(x,t)\, dx = \int_{\mathbb{R}^n} W(x,(4\pi)^{-1})\, dx = \int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = 1\,.$$

The proof is complete. □

The function $W(x,t)$ is called the *Gauss-Weierstrass kernel*. One can show that the function

$$w(x,t) = \int_{\mathbb{R}^n} W(x-y,t)f(y)\, dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

is the solution to the *heat equation* in the half-space

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta_x w & \text{on } \mathbb{R}^{n+1}_+, \\ w(x,0) = f(x), & x \in \mathbb{R}^n \end{cases}$$

under suitable assumptions about f.

The function $\Phi(x) = e^{-4\pi^2 t|x|^2}$ clearly satisfies the assumptions of Theorem 2.13. Hence we have

Theorem 2.15 (The Gauss-Weierstrass summability method)**.** *If $f \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} e^{-4\pi^2 t|\xi|^2}\, d\xi = \int_{\mathbb{R}^n} f(y) W(x-y,t)\, dy \to f$$

in $L^1(\mathbb{R}^n)$ as $t \to 0^+$.

Proof. The formula follows from Theorem 2.10 with $\varepsilon = 1$ and the fact that $W(y-x,t) = W(x-y,t)$. The convergence to f follows from Theorem 2.11 and Theorem 2.14(b,c). □

Corollary 2.16. *If both f and $\hat{f}$ are integrable*[2]*, then*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \quad a.e. \tag{2.3}$$

Proof. $M_t(x) = \int_{\mathbb{R}^n} f(y) W(x-y,t)\, dy$ converges to f in L^1 as $t \to 0^+$. Hence $M_{t_k} \to f$ a.e. for some sequence $t_k \to 0$. On the other hand integrability of $\hat{f}$ and the Lebesgue dominated convergence theorem yield

$$M_t(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} e^{-4\pi^2 t|\xi|^2}\, d\xi \to \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \quad \text{as } t \to 0^+$$

for all $x \in \mathbb{R}^n$. □

Corollary 2.17. *If $f_1, f_2 \in L^1(\mathbb{R}^n)$ and $\hat{f}_1 = \hat{f}_2$ on $\mathbb{R}^n$, then $f_1 = f_2$ a.e.*

Proof. Let $f = f_1 - f_2$. Then $\hat{f} = 0$ and hence the function M_ε from Theorem 2.13 equals zero. Since M_ε converges to f we conclude that $f = 0$ a.e. □

The following result provides another example of a function that satisfies the assumptions of Theorem 2.13.

Theorem 2.18. *Let $f(x) = e^{-2\pi|x|t}$, $t > 0$. Then*

(a)
$$P(x,t) := \hat{f}(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}},$$
where
$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

(b) *The function P has the following scaling property with respect to t: if $\varphi(x) = P(x,1)$, then $P(x,t) = \varphi_t(x)$.*

(c)
$$\int_{\mathbb{R}^n} P(x,t)\, dx = 1 \quad \textit{for all } t > 0.$$

[2]The assumption about integrability of $\hat{f}$ is very strong. Indeed, the equality (2.3) implies that f equals a.e. to a function in $C_0(\mathbb{R}^n)$.

The function $P(x,t)$ is called the *Poisson kernel*. Later we will prove that the function

$$u(x,t)=\int_{\mathbb{R}^n}P(x-y,t)f(y)\,dy$$

is a solution to the *Dirichlet problem* in the half-space

$$\begin{cases}\Delta_{(x,t)}u=0 & \text{on } \mathbb{R}^{n+1}_+,\\ u(x,0)=f(x), & x\in\mathbb{R}^n\end{cases}$$

under suitable assumptions about f.

By the same arguments as before we obtain.

Theorem 2.19 (The Abel summability method)**.** *If $f\in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i x\cdot\xi}e^{-2\pi|\xi|t}\,d\xi=\int_{\mathbb{R}^n}f(y)P(x-y,t)\,dy\to f$$

in $L^1(\mathbb{R}^n)$ as $t\to 0^+$.

The proof of Theorem 2.18 is substantially more difficult than that of Theorem 2.14. Since the formula for the Fourier transform involves the Γ function we need to recall its basic properties.

Definition. For $0<x<\infty$ we define

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt\,.$$

Theorem 2.20.

(a) $\Gamma(x+1)=x\Gamma(x)$ *for all* $0<x<\infty$.
(b) $\Gamma(n+1)=n!$ *for all* $n=0,1,2,3,\ldots$
(c) $\Gamma(1/2)=\sqrt{\pi}$.

Proof. (a) follows from the integration by parts. Since $\Gamma(1)=1$, (b) follows from (a) by induction. The substitution $t=s^2$ gives

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt=\int_0^\infty s^{2(x-1)}e^{-s^2}2s\,ds=2\int_0^\infty s^{2x-1}e^{-s^2}\,ds$$

and hence

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-s^2}\,ds=\sqrt{\pi}\,.$$

□

Lemma 2.21.

$$\int_0^{\pi/2}\sin^n\theta\,d\theta=\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}\quad \textit{for } n=1,2,3,\ldots$$

Proof. Denote the left hand side by a_n and the right hand side by b_n. Easy one time integration by parts shows that

$$a_{n+2} = (n+1)(a_n - a_{n+2}), \quad a_{n+2} = \frac{n+1}{n+2}\, a_n \,.$$

Also elementary properties of the Γ function show that

$$b_{n+2} = \frac{n+1}{n+2}\, b_n$$

and now it is enough to observe that $a_1 = 1 = b_1 = 1$, $a_2 = \pi/4 = b_2$. □

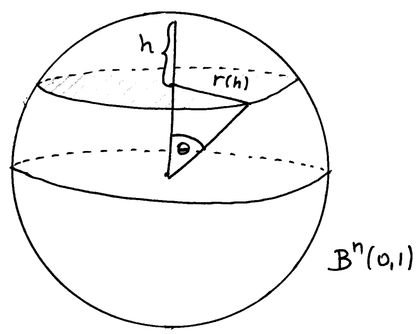
Lemma 2.22.

(a) *The volume of the unit ball in* $\mathbb{R}^n$ *equals*

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \,. \tag{2.4}$$

(b) *The* $(n-1)$*-dimensional measure of the unit sphere in* $\mathbb{R}^n$ *equals* $n\omega_n$

Proof.



It follows from the picture and the Fubini theorem that the volume of the upper half of the ball equals

$$\frac{1}{2}\omega_n = \int_0^1 \omega_{n-1} r(h)^{n-1}\, dh \,.$$

The substitution

$$h = 1 - \cos\theta, \quad dh = \sin\theta\, d\theta, \quad r(h) = \sin\theta$$

and Lemma 2.21 give

$$\frac{1}{2}\omega_n = \omega_{n-1}\int_0^{\pi/2} \sin^n\theta\, d\theta = \omega_{n-1}\, \frac{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \,,$$

so

$$\omega_n = \frac{2\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}\, \omega_{n-1} \,.$$

If

$$a_n = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}$$

then a direct computation shows that $a_1 = 2 = \omega_1$ and that a_n satisfies the same recurrence relationship as ω_n, so $\omega_n = a_n$ for all n. The second equality in (2.4) follows from Theorem 2.20(a).

(b) follows from the fact that the $(n-1)$-dimensional measure of a sphere of radius r equals to the derivative with respect to r of the volume of an n-dimensional ball of radius r. □

We will also need the following result.

Lemma 2.23. *For $\beta > 0$ we have*

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, e^{-\beta^2/(4u)}\, du\,.$$

Applying the theory of residues to the function $e^{i\beta z}/(1+z^2)$ one can easily prove that

$$e^{-\beta} = \frac{2}{\pi} \int_0^\infty \frac{\cos \beta x}{1+x^2}\, dx\,.$$

We also need an obvious identity

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u}\, du\,.$$

We have

$$\begin{aligned}
e^{-\beta} &= \frac{2}{\pi} \int_0^\infty \frac{\cos \beta x}{1+x^2}\, dx \\
&= \frac{2}{\pi} \int_0^\infty \cos \beta x \Big(\int_0^\infty e^{-u} e^{-ux^2}\, du \Big)\, dx \\
&= \frac{2}{\pi} \int_0^\infty e^{-u} \Big(\int_0^\infty e^{-ux^2} \cos \beta x\, dx \Big)\, du \\
&= \frac{2}{\pi} \int_0^\infty e^{-u} \Big(\frac{1}{2} \int_{-\infty}^\infty e^{-ux^2} e^{i\beta x}\, dx \Big)\, du \\
&= \frac{2}{\pi} \int_0^\infty e^{-u} \Big(\pi \underbrace{\int_{-\infty}^\infty e^{-4\pi^2 u y^2} e^{-2\pi i \beta y}\, dy}_{W(\beta,u)} \Big)\, du \\
&= \frac{2}{\pi} \int_0^\infty e^{-u} \Big(\frac{1}{2} \sqrt{\frac{\pi}{u}}\, e^{-\beta^2/(4u)} \Big)\, du \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, e^{-\beta^2/(4u)}\, du\,.
\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.18. (a) By a change of variables formula it suffices to prove the formula for $t = 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2\pi|x|} e^{-2\pi i x\cdot\xi}\,dx &= \int_{\mathbb{R}^n}\Big(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-4\pi^2|x|^2/(4u)}\,du\Big) e^{-2\pi i x\cdot\xi}\,dx \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\Big(\underbrace{\int_{\mathbb{R}^n} e^{-4\pi^2|x|^2/(4u)} e^{-2\pi i x\cdot\xi}\,dx}_{W(\xi,(4u)^{-1})}\Big)\,du \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\left(\sqrt{\frac{u}{\pi}}\right)^n e^{-u|\xi|^2}\,du \\
&= \frac{1}{\pi^{(n+1)/2}}\int_0^\infty e^{-u(1+|\xi|^2)} u^{(n-1)/2}\,du \\
&= \frac{1}{\pi^{(n+1)/2}}\frac{1}{(1+|\xi|^2)^{(n+1)/2}}\underbrace{\int_0^\infty e^{-s} s^{(n-1)/2}\,ds}_{\Gamma[(n+1)/2]}\,.
\end{aligned}$$

(b) is obvious.

(c) Because of the scaling property (b) it suffices to consider $t = 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} P(x,1)\,dx &= c_n\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+1)/2}} \\
&\overset{\text{polar}}{=} c_n\int_0^\infty\left(\int_{S^{n-1}(0,1)}\frac{d\sigma}{(1+r^2)^{(n+1)/2}}\right) r^{n-1}\,dr \\
&= c_n\, n\,\omega_n\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{(n+1)/2}}\,dr \\
&\overset{r=\tan\theta}{=} c_n\, n\,\omega_n\int_0^{\pi/2}\sin^{n-1}\theta\,d\theta \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}\, n\,\frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)}\,\frac{\pi^{1/2}\Gamma\left(\frac{n}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \\
&= 1\,.
\end{aligned}$$

The proof is complete. □

If $f, \hat{f} \in L^1(\mathbb{R}^n)$, then

$$f(x) = \int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \quad \text{a.e.} \tag{2.5}$$

by Corollary 2.16. The integral on the right hand side defines a continuous[3] function of $x \in \mathbb{R}^n$ and hence if in addition f is continuous, (2.5) holds everywhere. In particular we can apply the formula to

$$f(x) = e^{-4\pi^2 t|x|^2} \quad \text{and} \quad f(x) = e^{-2\pi t|x|}$$

[3] $C_0(\mathbb{R}^n)$.

which gives us

Corollary 2.24.

$$\int_{\mathbb{R}^n} W(\xi,t)e^{2\pi i x\cdot\xi}\, d\xi = e^{-4\pi^2 t|x|^2}\,,$$

$$\int_{\mathbb{R}^n} P(\xi,t)e^{2\pi i x\cdot\xi}\, d\xi = e^{-2\pi|x|t}$$

for all $x\in\mathbb{R}^n$.

The Weierstrass and Poisson kernels have the following semigroup property.

Corollary 2.25. *For* $t_1,t_2>0$ *we have*

$$\int_{\mathbb{R}^n} W(x-y,t_1)W(y,t_2)\, dy = W(x,t_1+t_2)\,,$$

$$\int_{\mathbb{R}^n} P(x-y,t_1)P(y,t_2)\, dy = P(x,t_1+t_2)$$

for all $x\in\mathbb{R}^n$. *In other words, if* $W_t(x)=W(x,t)$ *and* $P_t(x)=P(x,t)$, *then*

$$(W_{t_1} * W_{t_2})(x) = W_{t_1+t_2}(x)\,,$$

$$(P_{t_1} * P_{t_2})(x) = P_{t_1+t_2}(x)$$

for all $x\in\mathbb{R}^n$.

Proof. It follows from Corollary 2.24 with x replaced by $-x$ that

$$\hat{W}_t(x) = e^{-4\pi^2 t|x|^2}, \quad \hat{P}_t(x) = e^{-2\pi|x|t}\,.$$

Hence Theorem 2.7(b) yields

$$(W_{t_1} * W_{t_2})\hat{\ }(x) = \hat{W}_{t_1}(x)\hat{W}_{t_2}(x) = \hat{W}_{t_1+t_2}(x)$$

$$(P_{t_1} * P_{t_2})\hat{\ }(x) = \hat{P}_{t_1}(x)\hat{P}_{t_2}(x) = \hat{P}_{t_1+t_2}(x)$$

and the result follows from Corollary 2.17. □

As we know, if $f\in L^p(\mathbb{R}^n)$, $1\le p<\infty$ and $\varphi\in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\varphi=1$, then $f*\varphi_\varepsilon\to f$ in L^p. In particular

$$\int_{\mathbb{R}^n} f(y)W(x-y,t)\, dy \to f \quad \text{in } L^p \text{ as } t\to 0^+. \tag{2.6}$$

$$\int_{\mathbb{R}^n} f(y)P(x-y,t)\, dy \to f \quad \text{in } L^p \text{ as } t\to 0^+. \tag{2.7}$$

However, it is also interesting to investigate whether we have a.e. convergence. As we shall see this is true.

Theorem 2.26 (Lebesgue differentiation theorem)**.** *If $f \in L^1_{\rm loc}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \fint_{B(x,r)} f(y)\, dy = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x)$$

for a.e. $x \in \mathbb{R}^n$.

We proved this result in Analysis I.

Note that the Lebesgue differentiation theorem can be stated as follows: if $f \in L^1_{\rm loc}(\mathbb{R}^n)$, then $(f * \varphi_\varepsilon)(x) \to f(x)$ a.e., where $\varphi(x) = \omega_n^{-1}\chi_{B(0,1)}$. The integrals at (2.6) and (2.7) are also convolutions and hence it is an obvious guess that the Lebesgue differentiation theorem should play central role in the proof of pointwise convergence of these integrals.

As a consequence of Theorem 2.26 we have.

Corollary 2.27. *If $f \in L^1_{\rm loc}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0 \tag{2.8}$$

for a.e. $x \in \mathbb{R}^n$.

Exercise. *Prove the corollary.*[4]

DEFINITION. Points $x \in \mathbb{R}^n$ for which (2.8) is satisfied are called Lebesgue points of f.

DEFINITION. Let $\varphi \in L^1(\mathbb{R}^n)$. We say that Ψ is a *radially decreasing majorant of φ* if

(a) $\Psi(x) = \eta(|x|)$ for some[5] $\eta : [0,\infty) \to [0,\infty]$.
(b) η is decreasing.[6]
(c) $|\varphi(x)| \le \Psi(x)$ a.e.

Every $\varphi \in L^1(\mathbb{R}^n)$ has the least radially decreasing majorant. Indeed, if

$$\eta_0(t) = \operatorname*{ess\,sup}_{|y|\ge t} |\varphi(y)|$$

then η is decreasing (although it may be equal to infinity on some interval) and the function

$$\Psi_0(x) = \eta_0(|x|) = \operatorname*{ess\,sup}_{|y|\ge |x|} |\varphi(y)|$$

[4]Hint: Consider $\fint_{B(x,r)} |f(y) - \rho|\, dy$ for all rational ρ.

[5]Functions of this form, i.e. functions constant on spheres $S^{n-1}(0,r)$ are called *radially symmetric*.

[6]i.e. nonincreasing.

is the least radially decreasing majorant. Thus the existence of an integrable radially decreasing majorant Ψ of φ is equivalent with the condition that $\Psi_0 \in L^1$.

Theorem 2.28. *Let $\varphi \in L^1(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi = a$. Suppose that φ has an integrable radially decreasing majorant*

$$\Psi(x) = \eta(|x|) \in L^1(\mathbb{R}^n)\,.$$

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then

$$(f * \varphi_\varepsilon)(x) \to af(x)$$

whenever x is a Lebesgue point of f. In particular the Gauss-Weierstrass (2.6) and the Poisson (2.7) integrals converge to f a.e. as $t \to 0^+$.

Proof. We will prove the theorem under the additional assumption that η is absolutely continuous, but the result hods also without this assumption.

The integrability of Ψ gives a growth estimate for the function η. Indeed,

$$\int_{r/2\le|x|\le r} \Psi(x)\, dx \ge \eta(r)\left(\omega_n r^n - \omega_n\left(\frac{r}{2}\right)^n\right) = \omega_n\, \frac{2^n-1}{2^n}\, r^n\eta(r)\,.$$

The left hand side converges to 0 as $r \to 0$ or $r \to \infty$ so does the right hand side

$$\lim_{r\to 0} r^n\eta(r) = 0, \quad \lim_{r\to\infty} r^n\eta(r) = 0 \tag{2.9}$$

and hence the right hand side is bounded

$$r^n\eta(r) \le M \quad \text{for all } r > 0 \text{ and some } M > 0.$$

Fix a Lebesgue point x of f. Then for every $\gamma > 0$ there is $\delta > 0$ such that

$$\frac{1}{r^n}\int_{B(x,r)} |f(x-y) - f(x)|\, dy < \gamma \quad \text{provided } r \le \delta.$$

Using polar coordinates we can rewrite this inequality as

$$\frac{1}{r^n}\int_0^r s^{n-1}\underbrace{\Big(\int_{S^{n-1}(0,1)} |f(x-s\theta) - f(x)|\, d\sigma(\theta)\Big)}_{g(s)} ds < \gamma\,,$$

i.e.

$$G(r) = \int_0^r s^{n-1} g(s)\, ds < \gamma r^n \quad \text{provided } r \le \delta. \tag{2.10}$$

We have

$$\begin{aligned}
|(f*\varphi_\varepsilon)(x) - af(x)| &= \Big|\int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)\, dy\Big| \\
&\le \int_{|y|<\delta} |f(x-y) - f(x)|\Psi_\varepsilon(y)\, dy + \int_{|y|\ge\delta} |f(x-y) - f(x)|\Psi_\varepsilon(y)\, dy \\
&= I_1 + I_2\,.
\end{aligned}$$

We estimate I_1 as follows[7]

$$\begin{aligned}
I_1 &= \int_0^\delta s^{n-1}\Big(\int_{S^{n-1}(0,1)} |f(x-s\theta)-f(x)|\,d\sigma(\theta)\Big)\,\varepsilon^{-n}\eta(s/\varepsilon)\,ds \\
&= \int_0^\delta s^{n-1} g(s)\varepsilon^{-n}\eta(s/\varepsilon)\,ds \\
&= \varepsilon^{-n}\int_0^\delta G'(s)\eta(s/\varepsilon)\,ds \\
&= \varepsilon^{-n}G(s)\eta\left(\frac{s}{\varepsilon}\right)\Big|_0^\delta - \varepsilon^{-n-1}\int_0^\delta G(s)\eta'(s/\varepsilon)\,ds \\
&= G(s)s^{-n}\eta\left(\frac{s}{\varepsilon}\right)\left(\frac{s}{\varepsilon}\right)^n\Big|_0^\delta - \varepsilon^{-n}\int_0^{\delta/\varepsilon} G(\varepsilon t)\eta'(t)\,dt \\
&\overset{\eta'\leq 0}{\leq} \gamma M - \gamma\int_0^{\delta/\varepsilon} t^n\eta'(t)\,dt \\
&\overset{\eta'\leq 0}{\leq} \gamma M - \gamma\int_0^{\infty} t^n\eta'(t)\,dt \\
&\overset{\text{parts \& (2.9)}}{=} \gamma M + n\gamma\int_0^{\infty} t^{n-1}\eta(t)\,dt \\
&\overset{\text{polar}}{=} \gamma(M+\omega_n^{-1}\|\psi\|_1)\,.
\end{aligned}$$

The estimate for the integral I_2 is easier

$$I_2 \leq \int_{|y|\geq\delta} |f(x-y)|\psi_\varepsilon(y)\,dy + |f(x)|\int_{|y|\geq\delta}\psi_\varepsilon(y)\,dy = I_{21}+I_{22}.$$

Since

$$\int_{|y|\geq\delta}\psi_\varepsilon(y)\,dy = \int_{|y|\geq\delta/\varepsilon}\psi(y)\,dy \to 0 \quad \text{as } \varepsilon\to 0$$

we have $I_{22}\to 0$ as $\varepsilon\to 0$.

$$\begin{aligned}
I_{21} &\leq \|f\|_p\Big(\int_{|y|\geq\delta}\psi_\varepsilon^{p'}(y)\,dy\Big)^{1/p'} \\
&= \|f\|_p\Big(\int_{|y|\geq\delta}\psi_\varepsilon(y)\psi_\varepsilon^{p'/p}(y)\,dy\Big)^{1/p'} \\
&\leq \|f\|_p\Big(\sup_{|y|\geq\delta}\psi_\varepsilon(y)\Big)^{1/p}\Big(\int_{|y|\geq\delta}\psi_\varepsilon(y)\,dy\Big)^{1/p'} \\
&\leq \|f\|_p\left(\delta^{-n}\left(\frac{\delta}{\varepsilon}\right)^n\eta\left(\frac{\delta}{\varepsilon}\right)\right)^{1/p}\Big(\int_{|y|\geq\delta/\varepsilon}\psi(y)\,dy\Big)^{1/p'} \\
&\to 0 \qquad \text{as } \varepsilon\to 0
\end{aligned}$$

[7] If η is decreasing, but not absolutely continuous, then the integration by parts in the estimates below is not allowed. To overcome this difficulty one has to use the Stielties integral which allows to integrate by parts functions that are of bounded variation.

by (2.9). Therefore for $\varepsilon < \varepsilon_0$ the sum $I_1 + I_2$ is less than $2\gamma(M + \omega_n^{-1}\|\psi\|_1)$ which proves the theorem. □

2.4. **The Schwarz class and the Plancherel theorem.** We say that f belongs to the *Schwarz class* $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}_n$ if $f \in C^\infty(\mathbb{R}^n)$ and for all multiindices α, β

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| = p_{\alpha,\beta}(f) < \infty.$$

That means all derivatives of f rapidly decrease to zero as $|x| \to \infty$, faster than the inverse of any polynomial. Clearly $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}_n$, but also $e^{-|x|^2} \in \mathcal{S}_n$ so there are functions in the class $\mathcal{S}$ that have non compact support. $\{p_{\alpha,\beta}\}$ is a countable family of norms in $\mathcal{S}_n$ and we can use it to define a topology in $\mathcal{S}_n$. We say that a sequence (f_k) *converges to f in $\mathcal{S}_n$* if

$$\lim_{k\to\infty} p_{\alpha,\beta}(f_k - f) = 0$$

for all multiindices α, β. This convergence comes from a metric. Indeed, $d_{\alpha,\beta}(f, g) = p_{\alpha,\beta}(f - g)$ is a metric. If we arrange all these metrics in a sequence $d'_1, d'_2, \ldots$, then

$$d(f, g) = \sum_{k=1}^\infty 2^{-k} \frac{d'_k(f, g)}{1 + d'_k(f, g)}$$

is a metric in $\mathcal{S}_n$ such that $f_n \to f$ in $\mathcal{S}_n$ if and only if $f_n \to f$ in the metric d.

Recall that $(\tau_h f)(x) = f(h + x)$ for $h \in \mathbb{R}^n$.

Proposition 2.29. *The space $\mathcal{S}_n$ has the following properties.*

(a) *$\mathcal{S}_n$ equipped with the metric d is a complete metric space.*
(b) *$C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}_n$.*
(c) *If $\varphi \in \mathcal{S}_n$, then $\tau_h\varphi \to \varphi$ in $\mathcal{S}_n$ as $h \to 0$.*
(d) *The mapping*

$$\mathcal{S}_n \ni \varphi \mapsto x^\alpha D^\beta \varphi(x) \in \mathcal{S}_n$$

is continuous.
(e) *If $\varphi \in \mathcal{S}_n$, then*

$$\frac{\varphi(x + he_k) - \varphi(x)}{h} \to \frac{\partial\varphi}{\partial x_k}(x) \quad \text{as } h \to 0.$$

in the topology of $\mathcal{S}_n$.
(f) *If $\varphi, \psi \in \mathcal{S}_n$, then $\varphi * \psi \in \mathcal{S}_n$ and*

$$D^\alpha(\varphi * \psi) = (D^\alpha\varphi) * \psi = \varphi * (D^\alpha\psi)$$

for any multiindex α.

Exercise. *Prove the proposition.*

Theorem 2.30. *The Fourier transform is a continuous, one-to-one mapping of $\mathcal{S}_n$ onto $\mathcal{S}_n$ such that*

(a)
$$\left(\frac{\partial f}{\partial x_j}\right)^{\wedge}(\xi) = 2\pi i \xi_j \hat{f}(\xi),$$

(b)
$$(-2\pi i x_j f)\hat{}(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi)\,,$$

(c)
$$(f*g)\hat{} = \hat{f}\hat{g}, \quad (fg)\hat{} = \hat{f}*\hat{g}\,,$$

(d)
$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n}\hat{f}(x)g(x)\,dx\,,$$

(e)
$$f(x) = \int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\,.$$

Proof. We already proved (a), (b), first part of (c) and (d). The second part of (c) easily follows from its first part. Now we will prove that the Fourier transform is a continuous mapping from $\mathcal{S}_n$ into $\mathcal{S}_n$. Formulas (a) and (b) imply that

$$\xi^\alpha D^\beta \hat{f}(\xi) = C(D^\alpha(x^\beta f))\hat{}(\xi)$$

and hence

$$|\xi^\alpha D^\beta \hat{f}(\xi)| \leq C\|D^\alpha(x^\beta f)\|_1\,,$$
$$p_{\alpha,\beta}(\hat{f}) \leq C\|D^\alpha(x^\beta f)\|_1\,.$$

An application of the Leibnitz rule implies that $D^\alpha(x^\beta f)$ equals a finite sum of expressions of the form $x^{\beta_i}D^{\alpha_i}f$. Since

$$\begin{aligned}\|x^{\beta_i}D^{\alpha_i}f\|_1 &= \int_{\mathbb{R}^n}|(1+|x|^2)^n x^{\beta_i}D^{\alpha_i}f(x)|(1+|x|^2)^{-n}\,dx\\ &\leq C(n)\sup_{x\in\mathbb{R}^n}|(1+|x|^2)^n x^{\beta_i}D^{\alpha_i}f(x)| < \infty\end{aligned}$$

it follows that $\hat{f}\in\mathcal{S}_n$. One can also easily deduce that the mapping

$$\hat{}\,:\mathcal{S}_n\to\mathcal{S}_n$$

is continuous.

If $f\in\mathcal{S}_n$ than $\hat{f}\in\mathcal{S}_n$ and hence both f and $\hat{f}$ are integrable, so (e) follows from the inversion formula. This formula also shows that the Fourier transform applied four times is an identity on $\mathcal{S}_n$ and hence the Fourier transform is a bijection on $\mathcal{S}_n$. $\square$

Theorem 2.31 (Plancherel)**.** *The Fourier transform is an L^2 isometry on a dense subset $\mathcal{S}_n$ of L^2*

$$\|\hat{f}\|_2 = \|f\|_2, \quad f \in \mathcal{S}_n,$$

and hence it uniquely extends to an isometry of L^2

$$\|\hat{f}\|_2 = \|f\|_2, \quad f \in L^2(\mathbb{R}^n)\,.$$

Moreover for $f \in L^2(\mathbb{R}^n)$

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)e^{-2\pi i x\cdot\xi}\, dx$$

in the L^2 sense, i.e.

$$\Big\| \hat{f} - \int_{|x|<R} f(x)e^{-2\pi i x\cdot\xi}\, dx \Big\|_2 \to 0 \quad \textit{as } R\to\infty$$

and similarly

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$$

in the L^2 sense.

Proof. Given $f \in \mathcal{S}_n$ let $g = \bar{\hat{f}}$, so $\hat{g} = \bar{f}$. Indeed,

$$\hat{g}(\xi) = \int_{\mathbb{R}^n} \bar{\hat{f}}(x)e^{-2\pi i x\cdot\xi}\, dx = \overline{\int_{\mathbb{R}^n} \hat{f}(x)e^{2\pi i x\cdot\xi}\, dx} = \bar{f}(x)\,.$$

Hence Theorem 2.30(d) gives

$$\|f\|_2 = \int_{\mathbb{R}^n} f\bar{f} = \int_{\mathbb{R}^n} f\hat{g} = \int_{\mathbb{R}^n} \hat{f}g = \int_{\mathbb{R}^n} \hat{f}\bar{\hat{f}} = \|\hat{f}\|_2\,.$$

Thus the Fourier transform is an L^2 isometry on $\mathcal{S}_n$. Since $\mathcal{S}_n$ is a dense subset of L^2 it uniquely extends to an isometry of L^2. Now

$$L^1 \ni f\chi_{B(0,R)} \xrightarrow{L^2} f \quad \text{for } f \in L^2 \text{ as } R\to\infty$$

and hence

$$(f\chi_{B(0,R)})\hat{}\,(\xi) = \int_{|x|\leq R} f(x)e^{-2\pi i x\cdot\xi}\, dx \xrightarrow{L^2} \hat{f}(\xi)$$

as $R\to\infty$. Similarly

$$\int_{|\xi|<R} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi \xrightarrow{L^2} f(x)$$

as $R\to\infty$. □

Proposition 2.32. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f}(x)g(x)\, dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\, dx\,.$$

Proof. Approximate f and g in L^2 by functions in $\mathcal{S}_n$, apply Theorem 2.30(d) and pass to the limit. □

Consider the class $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ consisting of functions of the form $f = f_1 + f_2$, $f_1 \in L^1$, $f_2 \in L^2$. Then we define

$$\hat{f} = \hat{f}_1 + \hat{f}_2.$$

In order to show that the Fourier transform is well defined in the class $L^1 + L^2$ we need to show that it does not depend on the particular choice of the representation $f = f_1 + f_2$. Indeed, if we also have $f = g_1 + g_2$, $g_1 \in L^1$, $g_2 \in L^2$, then $f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$ and hence

$$\hat{f}_1 - \hat{g}_2 = (f_1 - g_1)\hat{} = (g_2 - f_2)\hat{} = \hat{g}_2 - \hat{f}_2,$$

$$\hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2\,.$$

It is an easy exercise (Problem 8) to show that

$$L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2,$$

and hence the Fourier transform is well defined on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$ and

$$\hat{}\colon L^p(\mathbb{R}^n) \to L^2(\mathbb{R}^n) + C_0(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2.$$

Later we will prove the Hausdorff-Young inequality which implies that

$$\hat{}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2.$$

2.5. **Tempered distributions.**

DEFINITION. The space $\mathcal{S}_n'$ of all continuous linear functionals on $\mathcal{S}_n$ is called the space of *tempered distributions*.

Here are examples:

1. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then

$$L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$$

defines a tempered distribution $L_f \in \mathcal{S}_n'$.

2. If μ is a measure of finite total variation, then

$$L_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi\,d\mu$$

defines a tempered distribution $L_\mu \in \mathcal{S}_n'$.

3. We say that a function f is a *tempered L^p function* if $f(x)(1+|x|^2)^{-k} \in L^p(\mathbb{R}^n)$ for some nonnegative integer k. If $p = \infty$ we call f a *slowly increasing function.* Then

$$L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,d\mu$$

defines a tempered distribution $L_f \in \mathcal{S}'_n$ for all $1 \le p \le \infty$. Note that slowly increasing functions are exactly measurable functions bounded by polynomials.

4. A *tempered measure* is a Borel measure μ such that

$$\int_{\mathbb{R}^n} (1+|x|^2)^{-k}\, d\mu < \infty$$

for some integer $k \ge 0$. As before $L_\mu \in \mathcal{S}'_n$.

5.

$$L(\varphi) = D^\alpha \varphi(x_0)$$

is a tempered distribution $L \in \mathcal{S}'_n$.

The distributions generated by a function or by a measure will often be denoted by $L_f(\varphi) = f[\varphi]$, $L_\mu(\varphi) = \mu[\varphi]$.

Suppose that $u \in \mathcal{S}'_n$. If there is a tempered L^p function f such that $u(\varphi) = f[\varphi]$ for $\varphi \in \mathcal{S}_n$, then we can identify u with the function f and simply write $u = f$. The identification is possible, because the function f is uniquely defined (up to a.e. equivalence). This follows from a well known result.

Lemma 2.33. *If $\Omega \subset \mathbb{R}^n$ is open and $f \in L^1_{\rm loc}(\Omega)$ satisfies $\int_\Omega f\varphi = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.*

Note that not every function $f \in C^\infty(\mathbb{R}^n)$ defines a tempered distribution, because it may happen that form some $\varphi \in \mathcal{S}_n$ the function $f\varphi$ is not integrable and hence the integral $f[\varphi] = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx$ does not make sense.

Theorem 2.34. *A linear functional on $\mathcal{S}_n$ is a tempered distribution if and only if there is a constant $C > 0$ and a positive integer m such that*

$$|L(\varphi)| \le C \sum_{|\alpha|,|\beta| \le m} p_{\alpha,\beta}(\varphi) \quad \textit{for all } \varphi \in \mathcal{S}_n.$$

Proof. If a linear functional L satisfies the given estimate, then clearly it is continuous on $\mathcal{S}_n$, so we are left with the proof of the converse implication. Let $L \in \mathcal{S}'_n$. We claim that there is a positive integer m such that $|L(\varphi)| \le 1$ for all

$$\varphi \in \Big\{ \varphi \in \mathcal{S}_n : \sum_{|\alpha|,|\beta| \le m} p_{\alpha,\beta}(\varphi) \le \frac{1}{m} \Big\} := N_m\, .$$

Suppose not. Then there is a sequence $\varphi_k \in \mathcal{S}_n$ such that $|L(\varphi_k)| > 1$ and

$$\sum_{|\alpha|,|\beta| \le k} p_{\alpha,\beta}(\varphi_k) \le \frac{1}{k}, \quad k = 1,2,3,\dots \tag{2.11}$$

Note that (2.11) implies that $\varphi_k \to 0$ in $\mathcal{S}_n$, so the inequality $|L(\varphi_k)| > 1$ contradicts continuity of L. This proves the claim. Denote

$$\|\varphi\| = \sum_{|\alpha|,|\beta|\le m} p_{\alpha,\beta}(\varphi)\,.$$

Observe that $\|\cdot\|$ is a norm. For an arbitrary $0 \neq \varphi \in \mathcal{S}_n$, $\tilde{\varphi} = \varphi/(m\|\varphi\|)$ satisfies $\|\tilde{\varphi}\| \le 1/m$, so $\tilde{\varphi} \in N_m$ and hence

$$|L(\varphi)| = m\|\varphi\||L(\tilde{\varphi})| \le m\|\varphi\|$$

which proves the theorem. □

For any function g on $\mathbb{R}^n$ we define $\tilde{g}(x) = g(-x)$. Then it easily follows from the Fubini theorem that for $u, \varphi, \psi \in \mathcal{S}_n$

$$\int_{\mathbb{R}^n} (u * \varphi)(x)\psi(x)\,dx = \int_{\mathbb{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\,dx\,. \tag{2.12}$$

Note that

$$\psi \mapsto \int_{\mathbb{R}^n} (u * \varphi)(x)\psi(x)\,dx, \quad \psi \in \mathcal{S}_n$$

and

$$\eta \mapsto \int_{\mathbb{R}^n} u(x)\eta(x)\,dx, \quad \eta \in \mathcal{S}_n$$

are tempered distributions. We denote them by $(u * \varphi)[\psi]$ and $u[\eta]$, so we can rewrite (2.12) as

$$(u * \varphi)[\psi] = u[\tilde{\varphi} * \psi]$$

Note that if $u \in \mathcal{S}_n'$ and $\varphi \in \mathcal{S}_n$, then $\psi \mapsto u[\tilde{\varphi} * \psi]$ is a tempered distribution.

Definition. If $u \in \mathcal{S}_n'$ and $\varphi \in \mathcal{S}_n$, then the *convolution of u and φ* is a tempered distribution defined by the formula

$$(u * \varphi)[\psi] := u[\tilde{\varphi} * \psi]\,.$$

The following result is left as an easy exercise.

Proposition 2.35. *If $u \in \mathcal{S}_n'$, $\varphi, \psi \in \mathcal{S}_n$, then*

$$(u * \varphi) * \psi = u * (\varphi * \psi)\,.$$

Theorem 2.36. *If $u \in \mathcal{S}_n'$ and $\varphi \in \mathcal{S}_n$, then the convolution $u * \varphi$ is the function f whose value at $x \in \mathbb{R}^n$ is*

$$f(x) = u[\tau_{-x}\tilde{\varphi}] = u[\varphi(x - \cdot)]\,.$$

Moreover $f \in C^\infty(\mathbb{R}^n)$ and f and all its derivatives are slowly increasing.

Proof. First we will prove that the function $f(x) = u[\tau_{-x}\tilde{\varphi}]$ is C^∞ and f as well as all its derivatives are slowly increasing. It follows from Theorem 2.29(b) that f is continuous. Observe that

$$\frac{\varphi(he_k - \cdot) - \varphi(-\cdot)}{h} = \frac{\tilde{\varphi}(\cdot - he_k) - \tilde{\varphi}(\cdot)}{h} \to -(\partial_k\tilde{\varphi})(\cdot) = (\partial_k\varphi)(-\cdot)$$

in the topology of $\mathcal{S}_n$ by Theorem 2.29(d), so for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\frac{\varphi(x+he_k-\cdot)-\varphi(x-\cdot)}{h} &= \tau_x\,\frac{\varphi(he_k-\cdot)-\varphi(-\cdot)}{h}\\ &\to \tau_x((\partial_k\varphi)(-\cdot)) = (\partial_k\varphi)(x-\cdot)\end{aligned}$$

in the topology of $\mathcal{S}_n$. Hence

$$\frac{f(x+he_k)-f(x)}{h} = u\left[\frac{\varphi(x+he_k-\cdot)-\varphi(x-\cdot)}{h}\right] \to u[(\partial_k\varphi)(x-\cdot)]\,.$$

Thus

$$\partial_k f(x) = u[(\partial_k\varphi)(x-\cdot)]$$

is continuous by Theorem 2.29(b). Since $\partial_k\varphi \in \mathcal{S}_n$, the function on the right hand side is of the same type as the one used to define f, so we can differentiate it again. Iterating this process we obtain

$$D^\alpha f(x) = u[(D^\alpha\varphi)(x-\cdot)]\,. \tag{2.13}$$

This implies $f \in C^\infty(\mathbb{R}^n)$. Since u is a tempered distribution, Theorem 2.34 gives the estimate

$$|f(x)| = |u[\tau_{-x}\tilde{\varphi}]| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\tau_{-x}\tilde{\varphi}) \leq C'(1+|x|)^m\,. \tag{2.14}$$

Indeed,

$$\begin{aligned}p_{\alpha,\beta}(\tau_{-x}\tilde{\varphi}) &= \sup_{z\in\mathbb{R}^n} |z^\alpha(D^\beta\tilde{\varphi})(z-x)| = \sup_{z\in\mathbb{R}^n} |(z+x)^\alpha(D^\beta\tilde{\varphi})(z)|\\ &\leq C\big(1+|x|^{|\alpha|}\big)\sup_{z\in\mathbb{R}^n}\big(1+|z|^{|\alpha|}\big)\,|D^\beta\tilde{\varphi}(z)| \leq C'(1+|x|)^m\,.\end{aligned}$$

Hence f is slowly increasing, and so it defines a tempered distribution. Since the derivatives of f satisfy (2.13) which is an expression of the same type as the one in the definition of f, we conclude that all derivatives $D^\alpha f$ are slowly increasing.

It remains to prove that

$$(u*\varphi)[\psi] = f[\psi] \quad \text{for all } \psi \in \mathcal{S}_n. \tag{2.15}$$

We have[8]

$$(u*\varphi)[\psi] = u[\tilde{\varphi}*\psi] = u\left[\int_{\mathbb{R}^n}\tilde{\varphi}(x-y)\psi(y)\,dy\right] = \heartsuit\,.$$

For each x we approximate the last integral by Riemann sums. The Riemann sums as a function of x belong to $\mathcal{S}_n$ and it is easy to see that they converge to $\tilde{\varphi}*\psi$ in the topology of $\mathcal{S}_n$. Hence from the linearity and continuity of u we have

$$\heartsuit = \int_{\mathbb{R}^n} u[\tilde{\varphi}(\cdot-y)]\psi(y)\,dy = \int_{\mathbb{R}^n} u[\tau_{-y}\tilde{\varphi}]\psi(y)\,dy = f[\psi]$$

which completes the proof.

[8] Valentine's day is soon.

We used the phrase "easy to see". This is what most of the textbooks write as an explanation of the convergence of Riemann sums to $\tilde{\varphi} * \psi$, but it is actually not that easy. The arguments needed in the proof of this fact are elementary, but quite technical and very unpleasant to write down. This is why in the textbooks the authors try to escape from the problem by saying "easy to see". To make the exposition self-contained we will explain this step with all details, but if we will need a similar argument in the future we will just say "easy to see".

First of all observe that is suffices to prove (2.15) under the assumption that $\varphi, \psi \in C_0^\infty$. Indeed, suppose we know (2.15) for compactly supported functions, but now $\varphi, \psi \in \mathcal{S}_n$. Let $C_0^\infty \ni \varphi_k \to \varphi$ in $\mathcal{S}_n$ and $C_0^\infty \ni \psi_\ell \to \psi$ in $\mathcal{S}_n$. Observe that it follows from the proof of (2.14) that all the functions $f_k = u[\tau_{-x}\tilde{\varphi}_k]$ have a common estimate independent of k

$$|f_k(x)| \leq C(1+|x|)^m$$

and hence $f_k \to f$ in $\mathcal{S}'_n$. It is also clear from the definition of the convolution that $u * \varphi_k \to u * \varphi$ in $\mathcal{S}'_n$, so

$$(u * \varphi)[\psi_\ell] = \lim_{k\to\infty} (u * \varphi_k)[\psi_\ell] = \lim_{k\to\infty} f_k[\psi_\ell] = f[\psi_\ell]\,.$$

Now letting $\ell \to \infty$ yields (2.15).

To approximate the integral $\int_{\mathbb{R}^n} \eta(y)\,dy$, $\eta \in C_0^\infty$ by Riemann sums, we fix a cube $Q = [-N, N]^n$ so large that $\operatorname{supp}\eta \subset Q$ and divide the cube into cubes $\{Q_{ki}\}_{i=1}^{m_k}$ of sidelength 2^{-k}. Denote the centers of these cubes by y_{ki}. Clearly

$$\sum_{i=1}^{m_k} \eta(y_{ki})|Q_{ki}| \to \int_{\mathbb{R}^n} \eta(y)\,dy \quad \text{as } k \to \infty,$$

and actually

$$\sum_{i=1}^{m_k} \int_{Q_{ki}} |\eta(y) - \eta(y_{ki})|\,dy \to 0 \quad \text{as } k \to \infty. \tag{2.16}$$

Suppose $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, so $\tilde{\varphi} * \psi \in C_0^\infty(\mathbb{R}^n)$. Hence for every $x \in \mathbb{R}^n$

$$w_k(x) = \sum_{i=1}^{m_k} \tilde{\varphi}(x - y_{ki})\psi(y_{ki})|Q_{ki}| \to \int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\psi(y)\,dy \quad \text{as } k \to \infty.$$

The functions $w_k(x)$ belong to $\mathcal{S}_n$.[9] We will prove that

$$w_k(x) \to \int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\psi(y)\,dy = (\tilde{\varphi} * \psi)(x) \quad \text{as } k \to \infty$$

in the topology of $\mathcal{S}_n$. First let us show that

$$\sup_{x\in\mathbb{R}^n} |w_k(x) - (\tilde{\varphi} * \psi)(x)| \to 0 \quad \text{as } k \to \infty.$$

[9]As finite linear combinations of functions from $\mathcal{S}_n$.

We have

$$
\begin{aligned}
|w_k(x)-(\tilde\varphi*\psi)(x)| &\leq \sum_{i=1}^{m_k}\int_{Q_{ki}}|\tilde\varphi(x-y_{ki})\psi(y_{ki})-\tilde\varphi(x-y)\psi(y)|\,dy\\
&\leq \|\tilde\varphi\|_\infty\sum_{i=1}^{m_k}\int_{Q_{ki}}|\psi(y_{ki})-\psi(y)|\,dy\\
&+ \sum_{i=1}^{m_k}\int_{Q_{ki}}|\tilde\varphi(x-y_{ki})-\tilde\varphi(x-y)|\,|\psi(y)|\,dy\\
&= I_1+I_2\,.
\end{aligned}
$$

$I_1\to 0$ as $k\to\infty$ by (2.16) and

$$
I_2\leq \sup_i\sup_{y\in Q_{ki}}|\tilde\varphi(x-y_{ki})-\tilde\varphi(x-y)|\int_{\mathbb{R}^n}|\psi(y)|\,dy\to 0\quad\text{as } k\to\infty,
$$

because $\tilde\varphi$ is uniformly continuous. Since

$$
D^\beta w_k(x)=\sum_{i=1}^{m_k}D^\beta\tilde\varphi(x-y_{ki})\psi(y_{ki})|Q_{ki}|\,,
$$

exactly the same argument as above shows that

$$
\sup_{x\in\mathbb{R}^n}|D^\beta w_k(x)-\underbrace{(D^\beta\tilde\varphi*\psi)(x)}_{D^\beta(\tilde\varphi*\psi)(x)}|\to 0\quad\text{as } k\to\infty.
$$

and hence

$$
p_{\alpha,\beta}(w_k-\tilde\varphi*\psi)\to 0\quad\text{as } k\to\infty,
$$

because $|x|^\alpha$ is bounded as the functions have compact support.

This proves that $w_k\to\tilde\varphi*\psi$ in $\mathcal{S}_n$. Since the function $f(x)=u[\tau_{-x}\tilde\varphi]$ is smooth, $f\psi\in C_0^\infty$ and hence

$$
\begin{aligned}
f[\psi] &= \int_{\mathbb{R}^n}\underbrace{u[\tilde\varphi(\cdot-y)]}_{f(y)}\psi(y)\,dy\\
&= \lim_{k\to\infty}\sum_{i=1}^{m_k}u[\tilde\varphi(\cdot-y_{ki})]\psi(y_{ki})|Q_{ki}|\\
&= \lim_{k\to\infty}u\left[\sum_{i=1}^{m_k}\tilde\varphi(\cdot-y_{ki})\psi(y_{ki})|Q_{ki}|\right]\\
&= \lim_{k\to\infty}u[w_k]=u[\tilde\varphi*\psi]\,.
\end{aligned}
$$

This time the proof is really complete. □

Note that formula (2.13) implies.

Theorem 2.37. *If $u\in\mathcal{S}_n'$ and $\varphi\in\mathcal{S}_n$, then for any multiindex α we have*

$$
D^\alpha(u*\varphi)(x)=(u*(D^\alpha\varphi))(x)\,.
$$

DEFINITION. The space $\mathcal{S}_n'$ is equipped with the *weak-$*$ convergence*, (called also *weak convergence*) i.e. we say that $u_k \to u$ in $\mathcal{S}_n'$ if

$$u_k(\psi) \to u(\psi) \quad \text{for every } \psi \in \mathcal{S}_n.$$

The following result seems obvious, but the proof requires the use of the Banach-Steinhaus theorem[10] (Corollary 9.2 from *Functional Analysis*).

Theorem 2.38. *If $u_k \to u$ is $\mathcal{S}_n'$ and $\varphi_k \to \varphi$ in $\mathcal{S}_n$, then $u_k(\varphi_k) \to u(\varphi)$.*

We leave details as an exercise.

This result together with Theorem 2.34 easily gives.

Theorem 2.39. *If $L_k \to L$ in $\mathcal{S}_n'$, then there is a constant $C > 0$ and a positive integer m such that*

$$\sup_k |L_k(\varphi)| \leq C \sum_{|\alpha|,|\beta| \leq m} p_{\alpha,\beta}(\varphi) \quad \textit{for all } \varphi \in \mathcal{S}_n.$$

Let $u \in \mathcal{S}_n'$. If $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi = 1$, then for every $\psi \in \mathcal{S}_n$, $\tilde{\varphi}_\varepsilon * \psi \to \psi$ in $\mathcal{S}_n$ (Why?), so

$$(u * \varphi_\varepsilon)[\psi] = u(\tilde{\varphi}_\varepsilon * \psi) \to u(\psi)\,.$$

Since $u_\varepsilon = u * \varphi_\varepsilon \in C^\infty$ we obtain a sequence of smooth functions that converge to u in $\mathcal{S}_n'$. If we take a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$, then one can easily prove that for $\psi \in \mathcal{S}_n$

$$\eta(x/k)\psi(x) \to \psi(x) \quad \text{in } \mathcal{S}_n \text{ as } k \to \infty.$$

Using this fact and Theorem 2.38 one can prove that

$$w_k(x) = \eta(x/k)(u * \varphi_{k^{-1}}) \in C_0^\infty$$

converges to u in $\mathcal{S}_n'$. This gives

Theorem 2.40. *$C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}_n'$, i.e. if $u \in \mathcal{S}_n'$, then there is a sequence $w_k \in C_0^\infty(\mathbb{R}^n)$ such that*

$$w_k[\psi] \to u(\psi) \quad \textit{for } \psi \in \mathcal{S}_n.$$

If $\varphi, \psi \in \mathcal{S}_n$, then the integration by parts gives

$$\int_{\mathbb{R}^n} D^\alpha \varphi(x)\psi(x)\,dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \varphi(x) D^\alpha \psi(x)\,dx\,.$$

For $h \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} (\tau_h \varphi)(x)\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)(\tau_{-h}\psi)(x)\,dx,$$

[10] At least I do not know how to avoid the Banach-Steinhaus theorem.

Moreover

$$\int_{\mathbb{R}^n} \tilde{\varphi}(x)\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\tilde{\psi}(x)\,dx\,,$$
$$\int_{\mathbb{R}^n} \hat{\varphi}(x)\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\hat{\psi}(x)\,dx\,.$$

If $\eta \in C^\infty$ is slowly increasing and all derivatives of η are slowly increasing, i.e. every derivative $D^\alpha\eta$ is bounded by a polynomial, then for $\psi \in \mathcal{S}_n$, $\eta\psi \in \mathcal{S}_n$ and the mapping $\psi \mapsto \eta\psi$ is continuous in $\mathcal{S}_n$. In particular $x^\alpha\psi(x) \in \mathcal{S}_n$. Moreover

$$\int_{\mathbb{R}^n} \big(\eta(x)\varphi(x)\big)\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\big(\eta(x)\psi(x)\big)\,dx\,.$$

This justifies the following definition.

Definition. If $u \in \mathcal{S}_n'$, then the *distributional partial derivative* $D^\alpha u$ is a tempered distribution defined by the formula

$$D^\alpha[\psi] = u[(-1)^{|\alpha|}D^\alpha\psi].$$

The translation $\tau_h u \in \mathcal{S}_n'$ is defined by

$$(\tau_h u)[\psi] = u[\tau_{-h}\psi].$$

The reflection $\tilde{u} \in \mathcal{S}_n'$ is

$$\tilde{u}[\psi] = u[\tilde{\psi}].$$

The Fourier transform $\hat{u} \in \mathcal{S}_n'$ is

$$\hat{u}[\psi] = u[\hat{\psi}]\,.$$

If $\eta \in C^\infty$ is slowly increasing and all derivatives of η are slowly increasing, then we define

$$(\eta u)[\psi] = u[\eta\psi]\,.$$

In particular

$$(x^\alpha u)[\psi] = u[x^\alpha\psi].$$

The Fourier transform on $\mathcal{S}_n'$ is often denoted by $\mathcal{F}u = \hat{u}$.

The formulas preceding the definition show that on the subclass $\mathcal{S}_n \subset \mathcal{S}_n'$ the partial derivative, the translation, the reflection, the Fourier transform and the multiplication by a function defined in the distributional sense coincide with those defined in the classical way.

If $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, in particular, if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then the classical Fourier transform coincides with the distributional one. That easily follows from Theorem 2.7(c) and Proposition 2.32.

The basic properties of the Fourier transform, distributional derivative and convolution in $\mathcal{S}_n'$ are collected in the next result whose easy proof is left to the reader.

Theorem 2.41. *The Fourier transform in a homeomorphism of $\mathcal{S}_n'$ onto itself.*[11] *Moreover for $u \in \mathcal{S}_n'$ and $\varphi \in \mathcal{S}_n$ we have*

(a) $(\hat{u})\hat{} = \tilde{u}$,
(b) $(u * \varphi)\hat{} = \hat{\varphi}\hat{u}$,
(c) $D^\alpha(u * \varphi) = D^\alpha u * \varphi = u * D^\alpha \varphi$,
(d) $(D^\alpha u)\hat{} = (2\pi i x)^\alpha \hat{u}$,
(e) $((-2\pi i x)^\alpha u)\hat{} = D^\alpha \hat{u}$.

Note that in the case (c), $u * \varphi \in \mathcal{S}_n'$ and $D^\alpha(u * \varphi)$ is understood in the distributional sense. On the other hand $u*\varphi \in C^\infty$ and Theorem 2.37 shows that the distributional derivative of $u * \varphi$ coincides with the classical one.

Let us compare the notion of distributional derivative in $\mathcal{S}_n'$ with the notion of weak derivative in Sobolev spaces.

Definition. Let $\Omega \subset \mathbb{R}^n$ be an open, $u, v \in L^1_{\rm loc}(\Omega)$ and let α be a multiindex. We say that $D^\alpha u = v$ in the *weak sense* if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega v\varphi = (-1)^{|\alpha|} \int_\Omega v D^\alpha \varphi \, .$$

Lemma 2.33 implies that the weak derivative $D^\alpha u$, if exists, is uniquely defined. If $u \in C^m(\Omega)$, then for $|\alpha| \leq m$ the the integration by parts gives

$$\int_\Omega D^\alpha v\varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

where $D^\alpha u$ is the classical partial derivative, so in this case the weak derivative coincides with the classical one.

It is an easy exercise to prove that if $f \in L^p$ is differentiable in L^p and $g = \partial f / \partial_{k_k}$ is the partial derivative in the L^p norm, then actually g is also a weak partial derivative of f.

If $u \in \mathcal{S}_n'$ and there is a tempered L^p function g such that

$$D^\alpha u[\psi] = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x)\psi(x)\, dx \quad \text{for } \psi \in \mathcal{S}_n.$$

then Lemma 2.33 shows that g is uniquely defined and we can identify $D^\alpha u = g$.

Definition. Let $1 \leq p \leq \infty$, m a positive integer and $\Omega \subset \mathbb{R}^n$ an open set. The Sobolev space $W^{m,p}(\Omega)$ is the set of functions $u \in L^p(\Omega)$ such that the weak partial derivatives of order less than or equal to m exist and belong to $L^p(\Omega)$. The space is equipped with the norm

$$\|u\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_p.$$

[11] With respect to the weak convergence in $\mathcal{S}_n'$.

It is an easy exercise to prove

Theorem 2.42. *$W^{m,p}(\Omega)$ is a Banach space.*

To compare Sobolev spaces with tempered distributions note that elements of $W^{m,p}(\mathbb{R}^n)$ belong to $\mathcal{S}_n'$ and the distributional derivatives of orders less than or equal m in the sense of $\mathcal{S}_n'$ coincide with the weak derivatives in the sense used to define the Sobolev space.

Note also that if $f \in L^p$ has partial derivatives $D^\alpha f$, $|\alpha| \leq m$ in the L^p norm, then $f \in W^{m,p}$ and the L^p derivatives coincide with the weak derivatives.

Using the notion of distributional derivative we can provide a new class of examples of a distributions in $\mathcal{S}_n'$. If f_α, $|\alpha| \leq m$ are slowly increasing functions and $a_\alpha \in \mathbb{C}$ for $|\alpha| \leq m$, then

$$u = \sum_{|\alpha|\leq m} a_\alpha D^\alpha (L_{f_\alpha}) \in \mathcal{S}_n' \,. \tag{2.17}$$

Surprisingly, every distribution in $\mathcal{S}_n'$ can be represented in the form (2.17). We will prove it now, but the proof will require some preparations.

First we will show how to solve, for each positive integer N, the partial differential equation

$$(I-\Delta)^N v = u\,, \tag{2.18}$$

where $u \in \mathcal{S}_n'$ is a given tempered distribution. A direct application of (2.1) shows that if $\psi \in \mathcal{S}_n$, then

$$\mathcal{F}\big((I-\Delta)^N\psi\big)(\xi) = (1+4\pi^2|\xi|^2)^N(\mathcal{F}\psi)(\xi)$$

and hence also

$$\mathcal{F}^{-1}\big((I-\Delta)^N\psi\big)(\xi) = (1+4\pi^2|\xi|^2)^N(\mathcal{F}^{-1}\psi)(\xi)\,. \tag{2.19}$$

Therefore the operator $(I-\Delta)^N$ can be represented as

$$(I-\Delta)^N = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^N\mathcal{F}\big) \tag{2.20}$$

Observe that the function $(1+4\pi^2|\xi|^2)^{-N}$ and all its derivatives are slowly increasing, so we can multiply tempered distributions by that function. For $u \in \mathcal{S}_n'$ we define

$$v = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}(\mathcal{F}u)\big) \in \mathcal{S}_n'\,.$$

We claim that v solves (2.18). Indeed for $\psi \in \mathcal{S}_n$ we have

$$(I-\Delta)^N v[\psi] = u\Big[\mathcal{F}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}^{-1}\big((I-\Delta)^N\psi\big)\big)\Big] = u[\psi]$$

by (2.19).

In view of (2.20) it is natural to define

$$(I-\Delta)^{-N} = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}\big)\,, \tag{2.21}$$

so

$$v = (I-\Delta)^{-N}u$$

is a solution to (2.18). The operator $(I-\Delta)^{-N}$ is called a *Bessel potential* and one can prove that it can be represented as an integral operator.[12]

We proved that every distribution $u \in \mathcal{S}_n'$ can be represented as

$$u = (I-\Delta)^{N}v\,, \tag{2.22}$$

where $v = (I-\Delta)^{-N}u \in \mathcal{S}_n'$. Now we will show that if N is sufficiently large, then v is actually a slowly increasing locally Lipschitz function, so (2.22) gives a representation of the form (2.17). We will also show that for any positive integer k we can find N so large that $v \in C^k(\mathbb{R}^n)$.

Theorem 2.43. *Suppose $u \in \mathcal{S}_n'$ satisfies the estimate*

$$|u(\varphi)| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi)\,.$$

If k is a nonnegative integer and $N > (n+m+k)/2$, then the tempered distribution

$$v = (I-\Delta)^{-N}u$$

has the following properties:

(a) *If $k = 0$, then v is a slowly increasing function.*
(b) *If $k = 1$, then v is a locally Lipschitz continuous slowly increasing function.*
(c) *If $k > 1$, then v is a slowly increasing function of the class C^{k-1}.*

Proof. We will need

Lemma 2.44. *If $P(x)$ is a polynomial of degree p and $N > (n+p)/2$, then all derivatives of*

$$f(x) = \frac{P(x)}{(1+|x|^2)^N}$$

belong to $L^1(\mathbb{R}^n)$.

Proof. Since $2N - p > n$, $f \in L^1(\mathbb{R}^n)$. We have

$$\frac{\partial f}{\partial x_i} = \frac{Q(x)}{(1+|x|^2)^{2N}}, \quad \deg Q = 2N + p - 1.$$

[12]Later we will carefully study Bessel potentials. We will find an explicit integral formula for the Bessel potential and we will show how to characterize Sobolev spaces in terms of Bessel potentials.

The function on the right hand side is of the same form as f. Since

$$2N > \frac{n+(2N+p-1)}{2}$$

we conclude that $\partial f/\partial x_i \in L^1$. The integrability of higher order derivatives follows by induction. $\Box$

We will prove that the distributional derivatives of v of orders $|\gamma| \leq k$ are slowly increasing functions. We have

$$(-1)^{|\gamma|} D^\gamma v[\psi] = u\big[\mathcal{F}\big((1+4\pi^2|x|^2)^{-N}\mathcal{F}^{-1}(D^\gamma\psi)\big)\big]\,.$$

Hence

$$\begin{aligned} |D^\gamma v[\psi]| &\leq C \sum_{|\alpha|,|\beta|\leq m} \sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta \Big(\mathcal{F}\big((1+4\pi^2|x|^2)^{-N}\mathcal{F}^{-1}(D^\gamma\psi)\big)\Big)\right| \\ &= C \sum_{|\alpha|,|\beta|\leq m} C_{\alpha,\beta,\gamma} \sup_{x\in\mathbb{R}^n} \left| \mathcal{F}\left(D^\alpha\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\,\mathcal{F}^{-1}(\psi)\right)\right)\right| \\ &\leq C' \sum_{|\alpha|,|\beta|\leq m} \left\| D^\alpha\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\,\mathcal{F}^{-1}(\psi)\right)\right\|_1 . \end{aligned}$$

Note that

$$\begin{aligned} & D^\alpha\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\,\mathcal{F}^{-1}(\psi)\right) \\ &= \sum_{\alpha_i+\beta_i=\alpha} \frac{\alpha!}{\alpha_1!\beta_i!} D^{\alpha_i}\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\right) D^{\beta_i}(\mathcal{F}^{-1}(\psi))\,. \end{aligned}$$

Since $\deg x^{\beta+\gamma} \leq m+k$ and $N > (n+m+k)/2$, Lemma 2.44 gives

$$D^{\alpha_i}\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\right) \in L^1(\mathbb{R}^n)\,,$$

so

$$\begin{aligned} |D^\gamma v[\psi]| &\leq C\sum_i \|\mathcal{F}^{-1}(x^{\beta_i}\psi)\|_\infty \\ &\leq C\sum_i \|x^{\beta_i}\psi\|_1 \\ &\leq C'\|(1+|x|^2)^{m/2}\psi(x)\|_1\,. \end{aligned}$$

This proves that for $|\gamma|\leq k$ the functional

$$\psi \mapsto (-1)^{|\gamma|}D^\gamma v[\psi]$$

is bounded on $L^1((1+|x|^2)^{m/2}\,dx)$. Thus there are functions $g_\gamma \in L^\infty$ such that

$$(-1)^{|\gamma|}D^\gamma v[\psi] = \int_{\mathbb{R}^n} \psi(x) g_\gamma(x)(1+|x|^2)^{m/2}\,dx\,,$$

i.e.

$$D^\gamma v = g_\gamma(x)(1+|x|^2)^{m/2}$$

in the distributional sense. In particular, if $\gamma = 0$, $v(x) = g_0(x)(1+|x|^2)^{m/2}$ is slowly increasing which proves (a). Moreover the distributional derivatives of orders $|\gamma| \le k$ are bounded on every bounded subset of $\mathbb{R}^n$, so v belongs to the Sobolev space $W^{k,\infty}$ on every bounded subset of $\mathbb{R}^n$ and the result follows from the following result that we state without proof.

Lemma 2.45. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, then the Sobolev space $W^{1,\infty}(\Omega)$ is the same as the space* $\mathrm{Lip}\,(\Omega)$ *of Lipschitz functions on Ω. More precisely every Lipschitz function on Ω belongs to $W^{1,\infty}(\Omega)$ and every function in $W^{1,\infty}(\Omega)$ equals a.e. to a Lipschitz function. If $k > 1$, then $W^{k,\infty}(\Omega) \subset C^{k-1}(\Omega)$ in the sense that every function in $W^{k,\infty}$ equals a.e. to a function in $C^{k-1}(\Omega)$.*

The proof is complete. □

We will investigate now properties of the Fourier transform of tempered distributions with compact support.

Lemma 2.46. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in \mathcal{S}_n'$. If $u(\varphi) = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $u(\varphi) = 0$ for all $\varphi \in \mathcal{S}_n$ such that* $\operatorname{supp}\varphi \subset \Omega$.

Proof. Let $\varphi \in \mathcal{S}_n$, $\operatorname{supp}\varphi \subset \Omega$. Let η be a cut-off function, i.e. $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$, then one can easily prove that $\eta(x/R)\varphi(x) \to \varphi(x)$ in $\mathcal{S}_n$ as $R \to \infty$. Since $\eta(x/R)\varphi(x) \in C_0^\infty(\Omega)$, the lemma follows. □

Definition. Let $u \in \mathcal{S}_n'$. The *support of* u ($\operatorname{supp} u$) is the intersection of all closed sets $E \subset \mathbb{R}^n$ such that

$$\varphi \in C_0^\infty(\mathbb{R}^n \setminus E) \quad \Rightarrow \quad u(\varphi) = 0.$$

Thus the support of u is the smallest closed set such that the distribution vanishes on C_0^∞ functions supported outside that set.

The lemma shows that we can replace $\varphi \in C_0^\infty(\mathbb{R}^n \setminus E)$ in the above definition by $\varphi \in \mathcal{S}_n$ with $\operatorname{supp}\varphi \subset \mathbb{R}^n \setminus E$.

Before we state the next result we need some facts about analytic and holomorphic functions in several variables.

Definition. We say that a function $f : \Omega \to \mathbb{C}$ defined in an open set $\Omega \subset \mathbb{R}^n$ is $\mathbb{R}$-*analytic*, if in a neighborhood on any point $x_0 \in \Omega$ it can be expanded as a convergent power series

$$f(x) = \sum_\alpha a_\alpha (x - x_0)^\alpha, \quad a_\alpha \in \mathbb{C}, \tag{2.23}$$

i.e. if in a neighborhood of any point f equals to its Taylor series.

We say that a function $f:\Omega\to\mathbb{C}$ defined in an open set $\Omega\subset\mathbb{C}^n$ is $\mathbb{C}$-*analytic* if in a neighborhood of any point $z_0\in\Omega$ it can be expanded as a convergent power series

$$f(z)=\sum_\alpha a_\alpha(z-z_0)^\alpha\,.$$

$\mathbb{R}^n$ has a natural embedding into $\mathbb{C}^n$, just like $\mathbb{R}$ into $\mathbb{C}$.

$$\mathbb{R}^n\ni x=(x_1,\ldots,x_n)\mapsto(x_1+i\cdot 0,\ldots,x_n+i\cdot 0)=x+i\cdot 0\in\mathbb{C}^n\,.$$

It is easy to see that an $\mathbb{R}$-analytic function f in $\Omega\subset\mathbb{R}^n$ extends to a $\mathbb{C}$-analytic function $\tilde f$ in an open set $\tilde\Omega\subset\mathbb{C}$, $\Omega\subset\tilde\Omega$. Namely, if f satisfies (2.23), we set

$$\tilde f(z)=\sum_\alpha a_\alpha(z-z_0)^\alpha\,.$$

On the other hand, if $\tilde f$ is $\mathbb{C}$-analytic in $\tilde\Omega\subset\mathbb{C}^n$, then the restriction f of $\tilde f$ to $\Omega=\tilde\Omega\cap\mathbb{R}^n$ is $\mathbb{R}$-analytic.

For example for any $\xi\in\mathbb{R}^n$, $f(x)=e^{x\cdot\xi}$ is $\mathbb{R}$-analytic and $\tilde f(x)=e^{z\cdot\xi}$ is its $\mathbb{C}$-analytic extension.

Definition. We say that a continuous function $f:\Omega\to\mathbb{C}$ defined in an open set $\Omega\subset\mathbb{C}^n$ is *holomorphic* if

$$\frac{\partial f}{\partial\overline{z}_i}=0\quad\text{for } i=1,2,\ldots,n.$$

It is easy to see that $\mathbb{C}$-analytic function are holomorphic, but the converse implication is also true.

Lemma 2.47 (Cauchy). *If f is holomorphic in*[13]

$$D^n(w,r)=D^1(w_1,r_1)\times\ldots\times D^1(w_n,r_n)$$

and continuous in the closure $\overline{D^n}(x,r)$, then

$$f(z)=\frac{1}{(2\pi i)^n}\int_{\partial D^1(w_1,r_1)}\cdots\int_{\partial D^1(w_n,r_n)}\frac{f(\xi)\,d\xi_1\ldots d\xi_n}{(\xi_1-z_1)\ldots(\xi_n-z_n)}\tag{2.24}$$

for all $z\in D^n(w,r)$.

Proof. The function f is holomorphic in each variable separately, so (2.24) follows from one dimensional Cauchy formulas and the Fubini theorem. □

Just like in the case of holomorphic functions of one variable one can prove that the integral on the right hand side of (2.24) can be expanded as a power series and hence defines a $\mathbb{C}$-analytic function. We proved.

Theorem 2.48. *A function f is holomorphic in $\Omega\subset\mathbb{C}^n$ if and only if it is $\mathbb{C}$-analytic.*

[13]Product of one dimensional discs.

Now we can state an important result about compactly supported distributions.

Theorem 2.49. *If $u \in \mathcal{S}'_n$ has compact support, then $\hat{u}$ is a slowly increasing C^∞ function and all derivatives of $\hat{u}$ are slowly increasing. Moreover $\hat{u}$ is $\mathbb{R}$-analytic on $\mathbb{R}^n$ and has a holomorphic extension to $\mathbb{C}^n$.*

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ in a neighborhood of $\operatorname{supp} u$. Then $u = \eta u$ in $\mathcal{S}'_n$ and hence for $\psi \in \mathcal{S}_n$ we have

$$\begin{aligned}\hat{u}[\psi] &= (\eta u)\hat{\ }[\psi] = u\left[\int_{\mathbb{R}^n} \eta(\cdot)\psi(x)e^{-2\pi i x\cdot(\cdot)}\, dx\right] \\ &= \int_{\mathbb{R}^n} u\left[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\right]\psi(x)\, dx\, .\end{aligned}$$

We could pass with u under the sign of the integral, because of an argument with approximation of the integral by Riemann sums.[14]

Note that the function

$$F(x_1,\ldots,x_n) = u\big[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]$$

is C^∞ smooth and

$$D^\alpha F(x_1,\ldots,x_n) = u\big[(-2\pi i(\cdot))^\alpha \eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]\, .$$

Indeed, we could differentiate under the sign of u because the corresponding difference quotients converge in the topology of $\mathcal{S}_n$. It also easily follows from Theorem 2.34 that F and all its derivatives are slowly increasing. Thus we may identify $\hat{u}$ with F, so $\hat{u} \in C^\infty$.

Moreover F has a holomorphic extension to $\mathbb{C}^n$ by the formula

$$F(z_1,\ldots,z_n) = u\big[\eta(\cdot)e^{-2\pi i z\cdot(\cdot)}\big]$$

so in particular $F(x_1,\ldots,x_n)$ is $\mathbb{R}$-analytic. □

Remark. Note that if $u \in \mathcal{S}'_n$ has compact support, then we can reasonably define $u[e^{-2\pi i x\cdot(\cdot)}]$ by the formula

$$u\big[e^{-2\pi i x\cdot(\cdot)}\big] := u\big[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]\, ,$$

since the right hand side does not depend on the choice of $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta = 1$ in a neighborhood of $\operatorname{supp} u$.

Theorem 2.50. *If $u \in \mathcal{S}'_n$ and $\operatorname{supp} u = \{x_0\}$, then there is an integer m and complex numbers a_α, $|\alpha| \le m$ such that*

$$u = \sum_{|\alpha|\le m} a_\alpha D^\alpha \delta_{x_0}\, .$$

[14] Compare with the proof of Theorem 2.36.

Proof. Without loss of generality we may assume that $x_0 = 0$. According to Theorem 2.34 the distribution u satisfies the estimate

$$|u(\varphi)| \le C \sum_{|\alpha|,|\beta|\le m} p_{\alpha,\beta}(\varphi)\,.$$

First we will prove that for $\varphi \in \mathcal{S}_n$ we have

$$D^\alpha\varphi(0) = 0 \quad \text{for } |\alpha| \le m \qquad \Longrightarrow \qquad u(\varphi) = 0\,. \tag{2.25}$$

Indeed, it follows from Taylor's formula that

$$\varphi(x) = O(|x|^{m+1}) \quad \text{as } x \to 0$$

and hence also

$$D^\beta\varphi(x) = O(|x|^{m+1-|\beta|}) \quad \text{as } x \to 0 \text{ for all } |\beta| \le m. \tag{2.26}$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function, i.e. $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le 1$, $\eta(x) = 0$ for $|x| \ge 2$ and define $\eta_\varepsilon(x) = \eta(x/\varepsilon)$. The estimate (2.26) easily implies that

$$p_{\alpha,\beta}(\eta_\varepsilon\varphi) \to 0 \quad \text{as } \varepsilon \to 0$$

for all $|\alpha|, |\beta| \le m$.[15] Note that $\varphi - \eta_\varepsilon\varphi = 0$ in a neighborhood of 0, so $u(\varphi - \eta_\varepsilon\varphi) = 0$. Hence

$$|u(\varphi)| \le |u(\varphi - \eta_\varepsilon\varphi)| + |u(\eta_\varepsilon\varphi)| \le 0 + \sum_{|\alpha|,|\beta|\le m} p_{\alpha,\beta}(\eta_\varepsilon\varphi) \to 0 \quad \text{as } \varepsilon \to 0.$$

This completes the proof of (2.25).

Let now $\psi \in \mathcal{S}_n$ be arbitrary and let

$$h(x) = \psi(x) - \sum_{|\alpha|\le m} \frac{D^\alpha\psi(0)}{\alpha!}\, x^\alpha\,.$$

Clearly

$$D^\alpha h(0) = 0 \quad \text{for } |\alpha| \le m. \tag{2.27}$$

We have

$$\psi(x) = \eta(x)\left(\sum_{|\alpha|\le m} \frac{D^\alpha\psi(0)}{\alpha!}\, x^\alpha\right) + \eta(x)h(x) + (1-\eta(x))\psi(x)\,.$$

Since $(1-\eta)\psi$ vanishes in a neighborhood of 0 we have $u((1-\eta)\psi) = 0$. The equality (2.27) implies that $\varphi = \eta h \in C_0^\infty(\mathbb{R}^n)$ satisfies the assumptions of

[15] Check it!

(2.25), so $u(\eta h) = 0$. Hence

$$\begin{aligned} u(\psi) &= u\Big[\eta(x)\Big(\sum_{|\alpha|\le m} \frac{D^\alpha\psi(0)}{\alpha!}\,x^\alpha\Big)\Big] \\ &= \sum_{|\alpha|\le m} \underbrace{(-1)^{|\alpha|} u\left[\frac{\eta(x)x^\alpha}{\alpha!}\right]}_{a_\alpha}\ \underbrace{(-1)^{|\alpha|}D^\alpha\psi(0)}_{D^\alpha\delta_0(\psi)}\,. \end{aligned}$$

The proof is complete. □

Corollary 2.51. *Let $u \in \mathcal{S}_n'$. If $\operatorname{supp}\hat{u} = \{\xi_0\}$, then u is a finite linear combination of functions $(-2\pi i x)^\alpha e^{2\pi i x\cdot\xi_0}$. In particular if $\operatorname{supp}\hat{u} = \{0\}$, then u is a polynomial.*

We leave the proof as an exercise.

Corollary 2.52. *If $u \in \mathcal{S}_n'$ satisfies $\Delta u = 0$, then u is a polynomial.*

Proof. We have

$$-4\pi^2|\xi|^2\hat{u} = (\Delta u)\hat{} = 0 \quad \text{in } \mathcal{S}_n'.$$

This implies that $\operatorname{supp}\hat{u} = \{0\}$, so u is a polynomial by Corollary 2.51. □

3. Interpolation of operators

If $1 \le p < q \le \infty$ and T is a linear operator such that $\|Tf\|_p \le A\|f\|_p$ and also $\|Tf\|_q \le B\|f\|_q$, then it turns out that for any $p < r < q$ there is a constant C such that $\|Tf\|_r \le C\|f\|_r$. To be more precise we should clarify on what space T is defined. For example we can assume that T is defined on $L^p + L^q$ or just on the class of all simple functions. The result seems very natural, but the proof is surprisingly difficult. Results of this type are called interpolation theorems. More generally *interpolation theorems* assume that an operator is bounded between some spaces and as a conclusion they give a larger class of spaces between which the operator is also bounded.

The result stated above can be proved by the real variable methods, but such methods require a lot of estimates and hence the result does not give the sharp estimate for the norm of the operator $T : L^r \to L^r$. To get the sharp estimate we need to use holomorphic functions. First we will prove an interpolation result (the Riesz-Thorin theorem) using holomorphic functions and this method is known as complex interpolation. Later we will prove another, more general result (the Marcinkiewicz theorem) using real methods. The Marcinkiewicz theorem is more general, but the estimate for the norm is not sharp. However, the Marcinkiewicz theorem being more general applies to the situations where the Riesz-Thorin theorem does not apply.

3.1. **Complex interpolation.** In the following result we assume that all L^r spaces consist of complex valued functions.

Theorem 3.1 (Riesz-Thorin). *Let (X,μ) and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0}$$
$$\|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$$

for all simple functions on X. Then for all $0<\theta<1$ we have

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p$$

for all simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,.$$

By density, T has a unique extension as a bounded operator from $L^p(\mu)$ to $L^q(\nu)$.

Before we prove the theorem we will show some applications. If $p_0 = q_0 < p_1 = q_1$, then $1/p = 1/q$ is a convex combination of $1/p_0$ and $1/p_1$, so p can be any number between p_0 and p_1, and hence $T : L^p \to L^p$ is bounded for any $p_0 < p < p_1$ as stated at the beginning of this section.

Although the Young inequality, Theorem 2.3 has an elementary proof it can also be concluded from the Riesz-Thorin theorem.

Proof of Thorem 2.3. Let $g \in L^r$, $1 \leq r \leq \infty$. The operator $Tf = f * g$ satisfies the estimate

$$\|Tf\|_r \leq \|g\|_r \|f\|_1 \quad \text{(Theorem 2.2)}$$

and

$$\|Tf\|_\infty \leq \|g\|_r \|f\|_{r'} \quad \text{(Hölder's inequality).}$$

Thus taking $q_0 = r$, $p_0 = 1$ and $q_1 = \infty$, $p_1 = r'$ we obtain from the Riesz-Thorin theorem

$$\|f*g\|_q = \|Tf\|_q \leq \|g\|_r^{1-\theta} \|g\|_r^{\theta} \|f\|_p = \|g\|_r \|f\|_p\,,$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'}, \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}, \quad \theta \in (0,1)\,,$$

i.e.

$$p = \frac{r}{r-\theta}, \quad q = \frac{r}{1-\theta}, \quad \theta \in (0,1)\,,$$

so p ranges from 1 to r', while q ranges from r to ∞ and $q^{-1} = p^{-1} + r^{-1} - 1$. $\square$

As we know, if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then $\hat{f} \in L^2(\mathbb{R}^n) + C_0(\mathbb{R}^n)$, but we also have

Theorem 3.2 (Hausdorff-Young)**.** *If* $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, *then*

$$\|\hat{f}\|_{p'} \le \|f\|_p \,.$$

Proof. Since

$$\|\hat{f}\|_\infty \le \|f\|_1, \quad \|\hat{f}\|_2 \le \|f\|_2 \,,$$

then the Riesz-Thorin theorem gives

$$\|\hat{f}\|_q \le 1^{1-\theta} 1^\theta \|f\|_p = \|f\|_p \,,$$

where

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \,, \quad \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} \,, \quad \theta \in (0,1) \,,$$

i.e.

$$q = \frac{2}{\theta}, \quad p = \frac{2}{2-\theta}, \quad \theta \in (0,1) \,,$$

so $p^{-1} + q^{-1} = 1$ and p is any exponent between 1 and 2. □

Remark. As we know (Problem 26), if $p > 2$, then there is a function $f \in L^p$ such that the distributional Fourier transform $\hat{f} \in \mathcal{S}_n'$ is not a function.

Proof of Theorem 3.1. Let

$$f = \sum_{k=1}^m a_k e^{i\alpha_k} \chi_{A_k}$$

be a simple function, where $a_k > 0$, $\alpha_k \in \mathbb{R}$ and A_k are pairwise disjoint subsets in X of finite measure. We want to estimate

$$\|Tf\|_q = \sup \left| \int_Y (Tf)(x) g(x) \, d\nu(x) \right| ,$$

where the supremum is over all simple functions

$$g = \sum_{j=1}^n b_j e^{i\beta_j} \chi_{B_j} \quad \text{with } \|g\|_{q'} \le 1 \,,$$

where $b_j > 0$, $\beta_j \in \mathbb{R}$ and B_j are pairwise disjoint subsets of Y of finite measure.

For $z \in \mathbb{C}$ let

$$f_z = \sum_{k=1}^m a_k^{P(z)} e^{i\alpha_k} \chi_{A_k} \,, \quad g_z = \sum_{j=1}^n b_j^{Q(z)} e^{i\beta_j} \chi_{B_j} \,,$$

where

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z, \qquad Q(z) = \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z \,.$$

$P(z)$ and $Q(z)$ are linear functions of z and hence the functions $a_k^{P(z)}$, $b_j^{Q(z)}$ are entire since $a_k, b_j > 0$. Observe that $P(\theta) = Q(\theta) = 1$, so $f_\theta = f$, $g_\theta = g$. Define

$$F(z) = \int_Y (Tf_z)(x) g_z(x)\, d\nu(x)\,.$$

In particular

$$F(\theta) = \int_Y (Tf) g\, d\nu\,.$$

Thus we extended the integral $\int_Y (Tf)g\, d\nu$ to a family of integrals depending on the parameter $z \in \mathbb{C}$ whose values form an entire function of z. Indeed, from the linearity of T we have

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_{B_j} T(\chi_{A_k})\, d\nu\,.$$

The integrals in the last formula are finite, so $F(z)$ is an entire function.

Let us now restrict z to be in the closed strip

$$\overline{S} = \{z \in \mathbb{C} :\ 0 \leq \operatorname{re} z \leq 1\}\,.$$

It easily follows from the formula that defines $F(z)$ that $|F(z)|$ is bounded on $\overline{S}$. Now we will obtain a more precise estimate for $|F(z)|$ on the boundary on that strip.

If $\operatorname{re} z = 0$, then

$$\|f_z\|_{p_0}^{p_0} = \sum_{k=1}^{m} |a_k^{P(z)}|^{p_0}\, \mu(A_k) = \sum_{k=1}^{m} a_k^p \mu(A_k) = \|f\|_p^p\,,$$

because the sets A_k are pairwise disjoint and $|a_k^{P(z)}| = a_k^{\operatorname{re} P(z)} = a_k^{p/p_0}$. Similarly

$$\|g_z\|_{q_0'}^{q_0'} = \|g\|_{q'}^{q'}\,.$$

Now Hölder's inequality and the assumptions about T give

$$|F(z)| \leq \|(Tf_z)\|_{q_0} \|g_z\|_{q_0'} \leq M_0 \|f_z\|_{p_0} \|g_z\|_{q_0'} = M_0 \|f\|_p^{p/p_0} \|g\|_{q'}^{q'/q_0'}\,.$$

If $\operatorname{re} z = 1$, then the same argument gives

$$\|f_z\|_{p_1}^{p_1} = \|f\|_p^p, \qquad \|g_z\|_{q_1'}^{q_1'} = \|g\|_{q'}^{q'}$$

and hence

$$|F(z)| \leq M_1 \|f\|_p^{p/p_1} \|g\|_{q'}^{q'/q_1'}\,.$$

The entire function $F(z)$ is bounded on $\overline{S}$ and we can use arguments from the theory of holomorphic functions to estimate $|F(z)|$ inside the strip by the estimates on the boundary. We need

Lemma 3.3 (The Hadamard three lines lemma)**.** *Let F be a holomorphic function in the open strip*

$$S = \{z \in \mathbb{C} : 0 < \operatorname{re} z < 1\},$$

continuous and bounded on the closure $\overline{S}$. If $|F(z)| \leq B_0$ when $\operatorname{re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{re} z = 1$, then

$$|F(z)| \leq B_0^{1-\operatorname{re} z} B_1^{\operatorname{re} z} \quad \textit{for } z \in \overline{S}.$$

Before we prove the lemma we show how to use it to complete the proof of the Riesz-Thorin theorem.

If $\operatorname{re} z = \theta$, the estimate for $F(z)$ on the boundary of $\overline{S}$ and the three lines lemma give

$$\begin{aligned} |F(z)| &\leq \Big(M_0 \|f\|_p^{p/p_0} \|g\|_{q'}^{q'/q'_0}\Big)^{1-\theta} \Big(M_1 \|f\|_p^{p/p_1} \|g\|_{q'}^{q'/q'_1}\Big)^{\theta} \\ &= M_0^{1-\theta} M_1^{\theta} \|f\|_p \|g\|_{q'} . \end{aligned}$$

In particular if $z = \theta$, then

$$\Big| \int_Y (Tf) g \, d\nu \Big| = |F(\theta)| \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p \|g\|_{q'}$$

and the result follows upon taking supremum over all simple functions g with $\|g\|_{q'} \leq 1$.

Proof of Lemma 3.3. Define the function

$$G(z) = F(z)(B_0^{1-z} B_1^z)^{-1} .$$

Note that $|G(z)| \leq 1$ on the boundary of $\overline{S}$, i.e. if $\operatorname{re} z = 0$ or $\operatorname{re} z = 1$. It suffices to prove that $|G(z)| \leq 1$ for all $z \in \overline{S}$.

Consider auxiliary functions $G_n(z) = G(z) e^{(z^2-1)/n}$ that will help us estimate $G(z)$.

Since F is bounded on $\overline{S}$ and $|B_0^{1-z} B_1^z|$ is bounded from below on $\overline{S}$, there is $M > 0$ such that $|G(z)| \leq M$ for $z \in \overline{S}$. We have

$$|G_n(x+iy)| \leq M e^{-y^2/n} e^{(x^2-1)/n} \leq M e^{-y^2/n} \quad \text{for } x+iy \in \overline{S}.$$

Thus $G_n(x+iy)$ converges uniformly to 0 in $0 \leq x \leq 1$ as $|y| \to \infty$. Let $z \in \overline{S}$. By the uniform convergence to 0 there is $y_0 > |\operatorname{im} z|$ such that

$$|G_n(x \pm iy_0)| \leq 1 \quad \text{for all } x \in [0,1].$$

Since $|G|$ is bounded by 1 on the boundary of the strip, also $|G_n|$ is bounded by 1 on that boundary. Thus $|G_n|$ is bounded by 1 on the boundary of the rectangle $[0,1] \times [-y_0, y_0]$, so $|G_n| \leq 1$ in the entire rectangle by the maximum principle. In particular $|G_n(z)| \leq 1$. Letting $n \to \infty$ we conclude that $|G(z)| \leq 1$. □

This completes the proof of the Riesz-Thorin theorem. □

3.2. **Real methods.** If $f \in L^p(\mu)$, $0<p<\infty$, then for any $t>0$

$$t^p\mu(\{x:|f(x)|>t\}) \leq \|f\|_p^p\,.$$

Hence

$$\sup_{t>0} t\,\mu(\{x:|f(x)|>t\})^{1/p} \leq \|f\|_p\,.$$

This suggests the following definition.

DEFINITION. Let (X,μ) be a measure space and $0<p<\infty$. The *Marcinkiewicz space*[16] $L^{p,\infty}(\mu)$ consists of all measurable functions on X such that

$$\|f\|_{p,\infty} = \sup_{t>0} t\,\mu(\{x:\,|f(x)|>t\})^{1/p} < \infty\,.$$

We also identify $L^{\infty,\infty} = L^\infty$ with $\|f\|_{\infty,\infty} = \|f\|_\infty$.

The previous argument gives

Proposition 3.4. *For $0<p\leq\infty$, $L^p(\mu)\subset L^{p,\infty}(\mu)$ and $\|f\|_p \leq \|f\|_{p,\infty}$.*

The inclusion $L^p\subset L^{p,\infty}$ is strict for $0<p<\infty$. For example $|x|^{-1}\in L^{1,\infty}(\mathbb{R})$, but $|x|^{-1}\notin L^1(\mathbb{R})$.

Exercise. *For any $0<p<\infty$ find $f\in L^{p,\infty}(\mathbb{R}^n)\setminus L^p(\mathbb{R}^n)$.*

Note that

$$\|kf\|_{p,\infty} = |k|\|f\|_{p,\infty} \quad \text{for } k\in\mathbb{C}$$

but

$$\|f+g\|_{p,\infty} \leq C_p(\|f\|_{p,\infty}+\|g\|_{p,\infty})$$

where $C_p=\max(2,2^{1/p})$, so $L^{p,\infty}$ is not a normed space, but a *quasi-normed* linear space when $0<p<\infty$.

We say that $f_n\to f$ in $L^{p,\infty}$ if $\|f-f_n\|_{p,\infty}\to 0$.

DEFINITION. We say that an operator T from a space of measurable functions into a space of measurable functions is *subadditive* if

$$|T(f_1+f_2)(x)| \leq |Tf_1(x)|+|Tf_2(x)| \quad \text{a.e.}$$

and

$$|T(kf)(x)| = |k||Tf(x)| \quad \text{for } k\in\mathbb{C}.$$

We say that a subadditive operator $T:L^p(\mu)\to L^q(\nu)$ is of *strong type* (p,q) if it is bounded, i.e.

$$\|Tf\|_q \leq C\|f\|_p$$

for some $C>0$ and all $f\in L^p(\mu)$ and it is of *weak type* (p,q) if

$$\|Tf\|_{q,\infty} \leq C\|f\|_p$$

for some $C>0$ and all $f\in L^p$. If $q=\infty$, then weak type (p,∞) is the same as the strong type (p,∞), because $L^{\infty,\infty}=L^\infty$.

[16] The Marcinkiewicz space is also called the weak L^p and denoted by weak-L^p or L^p_{w}.

Equivalently T is of weak type (p,q) if for all $t>0$

$$\mu(\{x:|Tf(x)|>t\}) \leq \left(\frac{C\|f\|_p}{t}\right)^q .$$

It follows immediately from Proposition 3.4 that operators of strong type are also of weak type.

It is important to observe that the L^p norm of a function f can be computed if we know measures of the level sets $\{x:\ |f(x)|>t\}$. Namely we have

Theorem 3.5 (Cavalieri's principle). *If μ is a σ-finite measure on X and $\Phi:[0,\infty)\to[0,\infty)$ is increasing, absolutely continuous and $\Phi(0)=0$, then*

$$\int_X \Phi(|f|)\,d\mu = \int_0^\infty \Phi'(t)\mu(\{|f|>t\})\,dt .$$

Proof. The result follows immediately from the equality

$$\int_X \Phi(|f(x)|)\,d\mu(x) = \int_X \int_0^{|f(x)|} \Phi'(t)\,dt\,d\mu(x)$$

and the Fubini theorem. □

Corollary 3.6. *If μ is a σ-finite measure on X and $0<p<\infty$, then*

$$\int_X |f|^p\,d\mu = p\int_0^\infty t^{p-1}\mu(\{|f|>t\})\,dt .$$

While the space $L^{p,\infty}$ is larger than L^p the difference is not big, because we have

Theorem 3.7 (Kolmogorov). *If $\mu(X)<\infty$, then $L^{p,\infty}(\mu)\subset L^q(\mu)$ for all $0<q<p$. Moreover*

$$\|f\|_q \leq 2^{1/q}\left(\frac{q}{p-q}\right)^{1/p}\mu(X)^{1/q-1/p}\|f\|_{p,\infty} .$$

Proof. Fix $t_0>0$. Using Corollary 3.6 and the estimates $\mu(\{|f|>t\})\leq \mu(X)$ for $0<t\leq t_0$ and $\mu(\{|f|>t\})\leq \|f\|^p_{p,\infty}t^{-p}$ for $t>t_0$ we get

$$\begin{aligned}\int_X |f|^q\,d\mu &\leq q\left(\int_0^{t_0} t^{q-1}\mu(X)\,dt + \|f\|^p_{p,\infty}\int_{t_0}^\infty t^{q-p-1}\,dt\right)\\ &= t_0^q\mu(X) + \frac{q}{p-q}\,t_0^{q-p}\|f\|^p_{p,\infty} .\end{aligned}$$

Then the result follows by choosing[17] $t_0=(q/(p-q))^{1/p}\mu(X)^{-1/p}\|f\|_{p,\infty}$. □

[17] This is a general trick. The right hand side is a sum of two expressions depending on t_0 and the inequality is true for any t_0, so we minimize the right hand side over t_0. Equivalently, we choose t_0 such that both summands on the right hand side are equal to each other.

Theorem 3.8 (Marcinkiewicz)**.** *Let (X,μ) and (Y,ν) be two measure spaces with σ-finite measures. Let $0 < p_0 \leq q_0 \leq \infty$, $0 < p_1 \leq q_1 \leq \infty$ and $q_0 \neq q_1$. Let T be a subadditive operator defined on the space $L^{p_0}(\mu) + L^{p_1}(\mu)$ taking values into the space of measurable functions on Y. Assume that*

$$\|Tf\|_{q_0,\infty} \leq M_0 \|f\|_{p_0} \quad \text{for } f \in L^{p_0}(\mu).$$
$$\|Tf\|_{q_1,\infty} \leq M_1 \|f\|_{p_1} \quad \text{for } f \in L^{p_1}(\mu).$$

Then for any $0 < \theta < 1$ there is a constant M depending on M_0, M_1, p_0, p_1, q_0, q_1 and θ only such that

$$\|Tf\|_q \leq M\|f\|_p \quad \text{for } f \in L^p(\mu)\,,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,.$$

Although the Marcinkiewicz theorem seems very similar to the Riesz-Thorin one, the two results are very different. Here the L^r spaces can consist of real valued functions, but in the Riesz-Thorin theorem they must be complex valued, because of the use of holomorphic functions. The price we pay for this is that the constant M is not as good as the one in the Riesz-Thorin theorem. Moreover we assume now that $p_0 \leq q_0$ and $p_1 \leq q_1$ and there was no such requirement in the previous interpolation result. On the other hand we allow exponents to be just greater than 0 and not greater than or equal to 1 and the operator needs only to be subadditive. However, the main difference which makes the Marcinkiewicz theorem so powerful is that we require the operator to be of weak type, while in the Riesz-Thorin theorem the operator had to be of strong type. Later we will see that in many situations it is possible to verify the weak type, so we can apply the Maricnkiewicz theorem, while in such situations the Riesz-Thorin theorem is useless.

We will prove only a special case of the Marcinkiewicz theorem, the case which is the most important in its applications.

Theorem 3.9 (Marcinkiewicz)**.** *Let (X,μ) and (Y,ν) be two measure spaces with σ-finite measures. Let T be a subadditive operator defined on $L^{p_0}(\mu) + L^{p_1}(\mu)$, where $0 < p_0 < p_1 \leq \infty$, and taking values into measurable functions on Y. Assume that there are constants $M_0, M_1 > 0$ such that*

$$\|Tf\|_{p_0,\infty} \leq M_0 \|f\|_{p_0} \quad \text{for } f \in L^{p_0}(\mu)$$
$$\|Tf\|_{p_1,\infty} \leq M_1 \|f\|_{p_1} \quad \text{for } f \in L^{p_1}(\mu).$$

Then for any $p_0 < p < p_1$

$$\|Tf\|_p \leq M\|f\|_p \quad \text{for } f \in L^p(\mu),$$

where

$$M = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{1/p} M_0^{1-\theta} M_1^{\theta}$$

if

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad 0<\theta<1\,.$$

Proof. Let $f\in L^p(\mu)$ and $t>0$. We decompose f as $f=f_0+f_1$, where

$$f_0=f\chi_{\{|f|>ct\}}\qquad f_1=f\chi_{\{|f|\le ct\}}\,.$$

The constant c will be chosen later. It is easy to see that $f_0\in L^{p_0}(\mu)$ and $f_1\in L^{p_1}(\mu)$. We have

$$|Tf(x)|\le|Tf_0(x)|+|Tf_1(x)|\,,$$

so

$$\{|Tf|>t\}\subset\{|Tf_0|>t/2\}\cup\{|Tf_1|>t/2\}$$

and hence

$$\nu(\{|Tf|>t\})\le\nu(\{|Tf_0|>t/2\})+\nu(\{|Tf_1|>t/2\})\,.$$

We will split the proof into two cases.

CASE 1: $p_1=\infty$. Choose $c=(2M_1)^{-1}$. We have

$$\|Tf_1\|_\infty\le M_1\|f_1\|_\infty\le M_1ct=\frac{t}{2}\,,$$

so

$$\nu(\{|Tf_1|>t/2\})=0\,.$$

By the weak (p_0,p_0) type we have

$$\nu(\{|Tf_0|>t/2\})\le\left(\frac{2M_0}{t}\,\|f_0\|_{p_0}\right)^{p_0}$$

and hence

$$\begin{aligned}
\|Tf\|_p^p &= p\int_0^\infty t^{p-1}\nu\big(\{|Tf|>t\}\big)\,dt\\
&\le p(2M_0)^{p_0}\int_0^\infty t^{p-p_0-1}\|f_0\|_{p_0}^{p_0}\,dt\\
&= p(2M_0)^{p_0}\int_0^\infty t^{p-p_0-1}\int_{\{x:|f(x)|>ct\}}|f(x)|^{p_0}\,d\mu(x)\,dt\\
&= p(2M_0)^{p_0}\int_X|f(x)|^{p_0}\int_0^{|f(x)|/c}t^{p-p_0-1}\,dt\,d\mu(x)\\
&= \frac{p}{p-p_0}\,(2M_0)^{p_0}\int_X|f(x)|^{p_0}\left(\frac{|f(x)|}{c}\right)^{p-p_0}\,d\mu(x)\\
&= \frac{p}{p-p_0}\,2^pM_0^{p_0}M_1^{p-p_0}\|f\|_p^p\,.
\end{aligned}$$

If

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{\infty},\quad 0<\theta<1\,,$$

then the above estimate reads as

$$\|Tf\|_p \leq 2\left(\frac{p}{p-p_0}\right)^{1/p} M_0^{1-\theta}M_1^{\theta}\|f\|_p\,.$$

Case 2: $p_1<\infty$. Now we have two inequalities arising from the weak type estimates

$$\nu(\{|Tf_0|>t/2\}) \leq \left(\frac{2M_0}{t}\,\|f_0\|_{p_0}\right)^{p_0},$$

$$\nu(\{|Tf_1|>t/2\}) \leq \left(\frac{2M_1}{t}\,\|f_1\|_{p_1}\right)^{p_1},$$

By an argument similar to the one used in Case 1 we have

$$\begin{aligned}\|Tf\|_p^p &\leq p\int_0^\infty t^{p-p_0-1}(2M_0)^{p_0}\|f_0\|_{p_0}^{p_0}\,dt\\ &+ p\int_0^\infty t^{p-p_1-1}(2M_1)^{p_1}\|f_1\|_{p_1}^{p_1}\,dt\\ &= \left(\frac{p}{p-p_0}\,(2M_0)^{p_0}c^{p_0-p}+\frac{p}{p-p_1}\,(2M_1)^{p_1}c^{p_1-p}\right)\|f\|_p^p\,.\end{aligned}$$

If we choose now c in a way that

$$(2M_0)^{p_0}c^{p_0}=(2M_1)^{p_1}c^{p_1}$$

the desired estimate will follow. □

3.3. **The Hardy-Littlewood maximal function.** As an immediate application of the Marcinkiewicz theorem we will prove the Hardy-Littlewood maximal theorem. For a locally integrable function $f\in L^1_{\rm loc}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal function* is defined by

$$\mathcal{M}f(x)=\sup_{r>0}⨍_{B(x,r)}|f(y)|\,dy,\quad x\in\mathbb{R}^n.$$

The operator $\mathcal{M}$ is not linear but it is subadditive.

Theorem 3.10. *If $f\in L^p(\mathbb{R}^n)$, $1\leq p\leq\infty$, then $\mathcal{M}f<\infty$ a.e. Moreover*

(a) *The operator $\mathcal{M}$ is of weak type $(1,1)$, i.e. for $f\in L^1(\mathbb{R}^n)$*

$$|\{x:\,\mathcal{M}f(x)>t\}|\leq\frac{5^n}{t}\int_{\mathbb{R}^n}|f|\quad\textit{for all } t>0. \tag{3.1}$$

(b) *If $f\in L^p(\mathbb{R}^n)$, $1<p\leq\infty$, then $\mathcal{M}f\in L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{M}f\|_p\leq 2\cdot 5^{n/p}\left(\frac{p}{p-1}\right)^{1/p}\|f\|_p\,.$$

Proof. It immediately follows from the definition of the maximal function that

$$\|\mathcal{M}f\|_\infty \le \|f\|_\infty .$$

Once we prove that the operator is on weak type $(1,1)$, i.e.[18]

$$\|\mathcal{M}f\|_{1,\infty} \le 5^n \|f\|_1 ,$$

the boundedness of $\mathcal{M}$ on L^p, i.e.

$$\|\mathcal{M}f\|_p \le 2\cdot 5^{n/p}\left(\frac{p}{p-1}\right)^{1/p} \|f\|_p$$

will follow from the Marcinkiewicz theorem. Thus we are left with the proof of the inequality (3.1). To this end we need an important covering lemma.

Theorem 3.11 (5r-covering lemma)**.** *Let $\mathcal{B}$ be a family of balls in a metric space such that* $\sup\{\operatorname{diam} B :\ B\in\mathcal{B}\}<\infty$*. Then there is a subfamily of pairwise disjoint balls $\mathcal{B}'\subset\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B .$$

If the metric space is separable, then the family $\mathcal{B}'$ is countable and we can arrange it as a sequence $\mathcal{B}'=\{B_i\}_{i=1}^\infty$, so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^\infty 5B_i .$$

Remark. Here $\mathcal{B}$ can be either a family of open balls or closed balls. In both cases proof is the same.

Proof. Let $\sup\{\operatorname{diam} B :\ B\in\mathcal{B}\}=R<\infty$. Divide the family $\mathcal{B}$ according to the diameter of the balls

$$\mathcal{F}_j=\{B\in\mathcal{B} : \frac{R}{2^j}<\operatorname{diam} B\le\frac{R}{2^{j-1}}\} .$$

Clearly $\mathcal{B}=\bigcup_{j=1}^\infty\mathcal{F}_j$. Define $\mathcal{B}_1\subset\mathcal{F}_1$ to be the maximal family of pairwise disjoint balls. Suppose the families $\mathcal{B}_1,\ldots,\mathcal{B}_{j-1}$ are already defined. Then we define $\mathcal{B}_j$ to be the maximal family of pairwise disjoint balls in

$$\mathcal{F}_j\cap\{B :\ B\cap B'=\emptyset \text{ for all } B'\in\bigcup_{i=1}^{j-1}\mathcal{B}_i\} .$$

Next we define $\mathcal{B}'=\bigcup_{j=1}^\infty\mathcal{B}_j$. Observe that every ball $B\in\mathcal{F}_j$ intersects with a ball in $\bigcup_{i=1}^j\mathcal{B}_j$. Suppose that $B\cap B_1\neq\emptyset$, $B_1\in\bigcup_{i=1}^j\mathcal{B}_i$. Then

$$\operatorname{diam} B\le\frac{R}{2^{j-1}}=2\cdot\frac{R}{2^j}\le 2\operatorname{diam} B_1$$

and hence $B\subset 5B_1$. □

[18]This inequality is equivalent to (3.1).

Let $f\in L^1(\mathbb{R}^n)$ and $E_t=\{x:\mathcal{M}f(x)>t\}$. For $x\in E_t$, there is $r_x>0$ such that

$$\fint_{B(x,r_x)}|f|>t\,,$$

so

$$|B(x,r_x)|<t^{-1}\int_{B(x,r_x)}|f|\,.$$

Observe that $\sup_{x\in E_t}r_x<\infty$, because $f\in L^1(\mathbb{R}^n)$. The family of balls $\{B(x,r_x)\}_{x\in E_t}$ forms a covering of the set E_t, so applying the $5r$-covering lemma there is a sequence of pairwise disjoint balls $B(x_i,r_{x_i})$, $i=1,2,\dots$ such that $E_t\subset\bigcup_{i=1}^\infty B(x_i,5r_{x_i})$ and hence

$$|E_t|\leq 5^n\sum_{i=1}^\infty|B(x_i,r_{x_i})|\leq\frac{5^n}{t}\sum_{i=1}^\infty\int_{B(x_i,r_{x_i})}|f|\leq\frac{5^n}{t}\int_{\mathbb{R}^n}|f|\,.$$

The proof is complete. □

As we shall see later the maximal function has many applications in analysis. Now we will show one such application.

3.4. **Theorem 2.28 revisited.** The following result is related to Theorem 2.28

Theorem 3.12. *Suppose that $\varphi\in L^1(\mathbb{R}^n)$ has an integrable radially decreasing majorant*

$$\Psi(x)=\eta(|x|)\in L^1(\mathbb{R}^n)\,.$$

Then for $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and all $x\in\mathbb{R}^n$

$$\left|\sup_{\varepsilon>0}(f*\varphi_\varepsilon)(x)\right|\leq\|\Psi\|_1\mathcal{M}f(x)\,. \tag{3.2}$$

Proof. The proof is similar to that of Theorem 2.28 and also now we will prove the result under the additional assumption that η is absolutely continuous. As in the proof of Theorem 2.28 we conclude from the integrability of Ψ that

$$\lim_{r\to 0}r^n\eta(r)=0,\qquad \lim_{r\to\infty}r^n\eta(r)=0\,. \tag{3.3}$$

Since both sides of the inequality (3.2) commute with translations we can assume that $x=0$. We can also assume that $\mathcal{M}f(0)<\infty$ as otherwise the inequality is obvious. We have

$$\begin{aligned}|(f*\varphi_\varepsilon)(0)| &\leq \int_{\mathbb{R}^n}|f(y)|\Psi_\varepsilon(y)\,dy\\ &= \int_0^\infty s^{n-1}\left(\int_{S^{n-1}}|f(s\theta)|\,d\sigma(\theta)\right)\varepsilon^{-n}\eta(s/\varepsilon)\,ds=\heartsuit\end{aligned}$$

Let

$$g(s)=\int_{S^{n-1}}|f(s\theta)|\,d\sigma(\theta)$$

and

$$G(r)=\int_0^r s^{n-1}g(s)\,ds=\int_{B(0,r)}|f(y)|\,dy\,.$$

Clearly

$$G(r)=\omega_n r^n \fint_{B(0,r)}|f(y)|\,dy\leq \omega_n r^n\mathcal{M}f(0)\,.$$

We have

$$\begin{aligned}\heartsuit &= \int_0^\infty G'(s)\varepsilon^{-n}\eta(s/\varepsilon)\,ds\\ &= \lim_{\substack{R\to\infty\\ r\to 0}} G(s)\varepsilon^{-n}\eta(s/\varepsilon)\Big|_r^R-\lim_{\substack{R\to\infty\\ r\to 0}}\int_r^R G(s)\varepsilon^{-n-1}\eta'(s/\varepsilon)\,ds=\diamondsuit \qquad (3.4)\end{aligned}$$

Since

$$G(s)\varepsilon^{-n}\eta(s/\varepsilon)\leq \omega_n\mathcal{M}f(0)(s/\varepsilon)^n\eta(s/\varepsilon)\,,$$

and the right hand converges to 0 as $s\to 0$ or $s\to\infty$ by (3.3), the fist limit at (3.4) equals 0 and hence

$$\begin{aligned}\heartsuit &= -\lim_{\substack{R\to\infty\\ r\to 0}}\int_r^R G(s)\varepsilon^{-n-1}\eta'(s/\varepsilon)\,ds\\ &= -\lim_{\substack{R\to\infty\\ r\to 0}}\int_{r/\varepsilon}^{R/\varepsilon} G(s\varepsilon)\varepsilon^{-n}\eta'(s)\,ds\\ &\leq -\lim_{\substack{R\to\infty\\ r\to 0}}\omega_n\mathcal{M}f(0)\int_{r/\varepsilon}^{R/\varepsilon} s^n\eta'(s)\,ds\\ &\overset{\text{parts \& (3.3)}}{=} n\omega_n\mathcal{M}f(0)\int_0^\infty s^{n-1}\eta(s)\,ds\\ &= \mathcal{M}f(0)\|\Psi\|_1\,.\end{aligned}$$

The last equality follows from the fact that $n\omega_n$ equals to the $(n-1)$ dimensional measure of the sphere $S^{n-1}(0,1)$ and the integration in spherical coordinates. □

4. Translation invariant operators

Definition. We say that a bounded operator $T:L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)$, $1\leq p,q\leq\infty$ *commutes with translations* if

$$T(\tau_h f)=\tau_h T(f)\quad\text{for } f\in L^p(\mathbb{R}^n) \text{ and } h\in\mathbb{R}^n.$$

The set of all such operators is denoted by $\mathcal{M}^{p,q}(\mathbb{R}^n)$. Clearly $\mathcal{M}^{p,q}(\mathbb{R}^n)$ is a normed linear space as a subspace of all bounded operators $B(L^p,L^q)$.

Theorem 4.1. *Suppose $T \in \mathcal{M}^{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$. Then there is a tempered distribution $u \in \mathcal{S}_n'$ such that*

$$T\varphi = u * \varphi \quad \text{for } \varphi \in \mathcal{S}_n.$$

Proof. We will need two lemmas.

Lemma 4.2. *If $T \in \mathcal{M}^{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$ and $\varphi \in \mathcal{S}_n$, then for all multiindices α, the distributional derivatives $D^\alpha(T\varphi)$ belong to L^q and*

$$D^\alpha(T\varphi) = T(D^\alpha\varphi)\,.$$

Lemma 4.3 (Sobolev)**.** *If $f \in W^{n+1,q}(\mathbb{R}^n)$, $1 \le q \le \infty$, then f equals a.e. to a continuous function (still denoted by f) such that*

$$|f(0)| \le C(n,q)\|f\|_{n+1,q}\,.$$

Assuming for a moment the two lemmas we will show how to prove the theorem. Let $T \in \mathcal{M}^{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$ and $\varphi \in \mathcal{S}_n$. According to Lemma 4.2, $T\varphi \in W^{m,q}(\mathbb{R}^n)$ for all m. In particular $T\varphi \in W^{n+1,q}(\mathbb{R}^n)$, so $T\varphi$ is a continuous function and the Sobolev lemma gives

$$\begin{aligned} |(T\varphi)(0)| &\le C(n,q) \sum_{|\beta|\le n+1} \|D^\beta(T\varphi)\|_q \\ &= C(n,q) \sum_{|\beta|\le n+1} \|T(D^\beta\varphi)\|_q \\ &\le C(n,q)\|T\|_{\mathcal{M}^{p,q}} \sum_{|\beta|\le n+1} \|D^\beta\varphi\|_p\,. \end{aligned}$$

We have

$$\begin{aligned} \|D^\beta\varphi\|_p &= \Big(\int_{\mathbb{R}^n} (1+|x|)^{(n+1)p}(1+|x|)^{-(n+1)p}|D^\beta\varphi(x)|^p\,dx\Big)^{1/p} \\ &\le \sup_{x\in\mathbb{R}^n}(1+|x|)^{n+1}|D^\beta\varphi(x)|\underbrace{\Big(\int_{\mathbb{R}^n}(1+|x|)^{-(n+1)p}\,dx\Big)^{1/p}}_{<\infty} \\ &\le C\sum_{|\alpha|\le n+1} p_{\alpha,\beta}(\varphi)\,, \end{aligned}$$

so

$$|(T\varphi)(0)| \le C \sum_{|\alpha|,|\beta|\le n+1} p_{\alpha,\beta}(\varphi)\,.$$

Thus

$$v(\varphi) = (T\varphi)(0)$$

defines a tempered distribution $v \in \mathcal{S}_n'$. Let $u = \tilde{v}$. We have

$$\begin{aligned} (u*\varphi)(x) &= (\tilde{v}*\varphi)(x) = \tilde{v}[\tau_{-x}\tilde{\varphi}] = v[\tau_x\varphi] \\ &= \big(T(\tau_x\varphi)\big)(0) = \big(\tau_x(T\varphi)\big)(0) = (T\varphi)(x)\,. \end{aligned}$$

We are left with the proofs of Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. Let $\varphi, \psi \in \mathcal{S}_n$. $T\varphi \in L^q$, so the distributional derivative $\partial_j(T\varphi) \in \mathcal{S}_n'$ is well defined. Since

$$\frac{\varphi(x) - \varphi(x - he_j)}{h} \to \partial_j \varphi(x) \quad \text{as } h \to 0$$

in the topology of $\mathcal{S}_n$ we have

$$\begin{aligned}
\partial_j(T\varphi)[\psi] &= -\int_{\mathbb{R}^n} (T\varphi)(x)\partial_j\psi(x)\,dx \\
&= -\lim_{h\to 0}\int_{\mathbb{R}^n} (T\varphi)(x)\frac{\psi(x+he_j)-\psi(x)}{h}\,dx \\
&= \lim_{h\to 0}\int_{\mathbb{R}^n} \frac{T\varphi(x)-(T\varphi)(x-he_j)}{h}\,\psi(x)\,dx \\
&= \lim_{h\to 0}\int_{\mathbb{R}^n} \frac{T\varphi(x)-(\tau_{-he_j}T)\varphi(x)}{h}\,\psi(x)\,dx \\
&= \lim_{h\to 0}\int_{\mathbb{R}^n} \frac{T\varphi(x)-T(\tau_{-he_j}\varphi)(x)}{h}\,\psi(x)\,dx \\
&= \lim_{h\to 0}\int_{\mathbb{R}^n} T\left(\frac{\varphi(\cdot)-\varphi(\cdot - he_j)}{h}\right)(x)\,\psi(x)\,dx \\
&= \int_{\mathbb{R}^n} T(\partial_j\varphi)(x)\psi(x)\,dx\,.
\end{aligned}$$

Thus

$$\partial_j(T\varphi) = T(\partial_j\varphi)$$

and by induction we have

$$D^\alpha(T\varphi) = T(D^\alpha\varphi) \in L^q(\mathbb{R}^n)\,.$$

The proof is complete. □

Proof of Lemma 4.3. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function[19] and let $\eta_R(x) = \eta(x/R)$. Then $\eta_R f \in L^1$. Since the weak derivatives satisfy the Leibniz rule

$$D^\alpha(\eta_R f) = \sum_{\beta_i+\gamma_i=\alpha} \frac{\alpha!}{\beta_i!\gamma_i!}\, D^{\beta_i}\eta_R D^{\gamma_i} f \tag{4.1}$$

and $D^{\beta_i}\eta_R \in C_0^\infty$ we conclude that $\eta_R f \in W^{n+1,1}(\mathbb{R}^n)$. The elementary inequality

$$1 \le C(n)(1+|x|)^{-(n+1)} \sum_{|\alpha|\le n+1} |(-2\pi i x)^\alpha|$$

[19] $0 \le \eta \le 1$, $\eta(x) = 1$ if $|x| \le 1$, $\eta(x) = 0$ if $|x| \ge 2$.

gives

$$\begin{aligned}|(\eta_R f)\hat{}(x)| &\leq C(n)(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \left|(-2\pi i x)^\alpha (\eta_R f)\hat{}(x)\right| \\ &\leq C(n)(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \left\|(D^\alpha\big(\eta_R f)\hat{}\right\|_\infty \\ &\leq C(n)(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \|D^\alpha(\eta_R f)\|_1 \\ &\leq C(n,R)(1+|x|)^{-(n+1)} \sum_{|\alpha|\leq n+1} \|D^\alpha f\|_q\,,\end{aligned}$$

where in the last inequality we applied (4.1) and Hölder's inequality. Integrating both sides with respect to x we have

$$\|(\eta_R f)\hat{}\|_1 \leq C(n,R)\|f\|_{n+1,q}\,.$$

Since $\eta_R f \in L^1$ and $(\eta_R f)\hat{} \in L^1$ we conclude that $\eta_R f \in C_0$. Since $f(x) = (\eta_R f)(x)$ for $|x| \leq R$ continuity of f on $\mathbb{R}^n$ follows. Moreover the inversion formula gives

$$|f(0)| = |(\eta_R f)(0)| = \Big|\int_{\mathbb{R}^n} (\eta_R f)\hat{}\Big| \leq C(n,R)\|f\|_{n+1,q}\,.$$

This completes the proof of Lemma 4.3 and hence that of Theorem 4.1. □

If $1 \leq p,q,r \leq \infty$ and $q^{-1} = p^{-1} + r^{-1} - 1$, then according to Young's inequality (Theorem 2.3),

$$\|f * g\|_q \leq \|f\|_p \|g\|_r\,,$$

so $Tf = f*g$ defines an operator in $\mathcal{M}^{p,q}$. Observe that $q \geq p$ and if r ranges from $r=1$ to $r = p' = p/(p-1)$, then q can be any exponent $p \leq q \leq \infty$. The fact that q has to be greater or equal to p is a very general one.

Theorem 4.4 (Hörmander). *If $1 \leq q < p < \infty$, then $\mathcal{M}^{p,q}(\mathbb{R}^n) = \{0\}$.*

Proof. We will need the following result which is interesting on its own.

Lemma 4.5. *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then*

$$\lim_{|h|\to\infty} \|\tau_h f + f\|_p = 2^{1/p}\|f\|_p\,.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\|f-\varphi\|_p < \varepsilon$. Then[20]

$$\big|\|\tau_h f + f\|_p - \|\tau_h\varphi + \varphi\|_p\big| \leq \|\tau_h(f-\varphi) + (f-\varphi)\|_p < 2\varepsilon\,.$$

If $|h|$ is large enough, then the supports of φ and $\tau_h\varphi$ are disjoint and hence

$$\|\tau_h\varphi + \varphi\|_p = 2^{1/p}\|\varphi\|_p\,.$$

[20] $\big|\|x\| - \|y\|\big| \leq \|x-y\|$.

Thus

$$\limsup_{|h|\to\infty} \big| \|\tau_h f + f\|_p - 2^{1/p}\|\varphi\|_p \big| \le 2\varepsilon\,.$$

Since $\varepsilon > 0$ can be arbitrary the lemma follows. □

Now we can complete the proof of the theorem. For $f \in L^p(\mathbb{R}^n)$ we have

$$\|\tau_h(T(f)) + T(f)\|_q = \|T(\tau_h f + f)\|_q \le \|T\|_{\mathcal{M}^{p,q}}\|\tau_h f + f\|_p\,.$$

Letting $|h| \to \infty$ the lemma yields

$$2^{1/q}\|T(f)\|_q \le \|T\|_{\mathcal{M}^{p,q}} 2^{1/p}\|f\|_p\,,$$

so

$$\|T\|_{\mathcal{M}^{p,q}} \le \|T\|_{\mathcal{M}^{p,q}} 2^{1/p-1/q}$$

which implies $T = 0$. □

The same argument shows that the only translation invariant operator $T : C_0(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 \le q < \infty$ is the zero operator.

Theorem 4.6. *Let $1 \le p \le q \le \infty$, $p < \infty$, $q > 1$ and $T \in \mathcal{M}^{p,q}(\mathbb{R}^n)$. Then*

$$T : L^p \cap L^{q'} \to L^{p'}$$

is a bounded operator, so it uniquely extends to $T \in \mathcal{M}^{q',p'}$. Moreover

$$\|T\|_{\mathcal{M}^{q',p'}} = \|T\|_{\mathcal{M}^{p,q}}\,.$$

In other words, we have isometric identification

$$\mathcal{M}^{p,q}(\mathbb{R}^n) = \mathcal{M}^{q',p'}(\mathbb{R}^n)\,.$$

Proof. Let $g \in L^{q'}$. Then

$$L^p \ni f \mapsto \int_{\mathbb{R}^n} T(f)g$$

is a bounded linear functional on L^p, so there is a unique function $T^*g \in L^{p'}$ such that

$$\int_{\mathbb{R}^n} T(f)g = \int_{\mathbb{R}^n} fT^*g\,.$$

In other words $T^* : L^{q'} \to L^{p'}$ is the adjoint operator. It is easy to see that T^* is translation invariant, so $T^* \in \mathcal{M}^{q',p'}$ and thus

$$\|T^*\|_{\mathcal{M}^{q',p'}} = \|T\|_{\mathcal{M}^{p,q}}$$

as the norm of the adjoint operator equals to the norm of the operator. Indeed,

$$\begin{aligned} \|T^*\|_{\mathcal{M}^{q',p'}} &= \sup_{\|g\|_{q'}\le 1} \|T^*g\|_{p'} = \sup_{\|g\|_{q'}\le 1}\sup_{\|f\|_p\le 1} \int_{\mathbb{R}^n} fT^*g \\ &= \sup_{\|f\|_p\le 1}\sup_{\|g\|_{q'}\le 1} \int_{\mathbb{R}^n} T(f)g = \|T\|_{\mathcal{M}^{p,q}}\,. \end{aligned}$$

Let $u \in \mathcal{S}_n'$ be such that

$$T\varphi = u * \varphi \quad \text{for } \varphi \in \mathcal{S}_n.$$

For $\varphi, \psi \in \mathcal{S}_n$ we have

$$\begin{aligned}(T^*\varphi)[\psi] &= (T\psi)[\varphi] = (u*\psi)[\varphi] = u[\tilde{\psi} * \varphi] \\ &= \tilde{u}[\psi * \tilde{\varphi}] = \tilde{u}[\tilde{\varphi} * \psi] = (\tilde{u} * \varphi)[\psi]\,,\end{aligned}$$

i.e.

$$T^*\varphi = \tilde{u} * \varphi \quad \text{for } \varphi \in \mathcal{S}_n.$$

Since

$$T^*\varphi = \tilde{u} * \varphi = (u * \tilde{\varphi})\tilde{} = (T\tilde{\varphi})\tilde{}$$

we see that $T^* \in \mathcal{M}^{q',p'}$ implies that $T \in \mathcal{M}^{q',p'}$ and

$$\|T\|_{\mathcal{M}^{q',p'}} = \|T^*\|_{\mathcal{M}^{q',p'}} = \|T\|_{\mathcal{M}^{p,q}}\,.$$

Thus $\mathcal{M}^{p,q} \subset \mathcal{M}^{q',p'}$ isometrically, but the same argument applied to $\mathcal{M}^{q',p'}$ in place of $\mathcal{M}^{p,q}$ gives the opposite inclusion, so $\mathcal{M}^{p,q} = \mathcal{M}^{q',p'}$ isometrically. □

Theorem 4.7. $T \in \mathcal{M}^{1,1}(\mathbb{R}^n)$ *if and only if*

$$(Tf)(x) = (f * \mu)(x) = \int_{\mathbb{R}^n} f(x-y)\, d\mu(y), \quad f \in L^1(\mathbb{R}^n)$$

for some complex-valued measure of finite total variation $\mu \in \mathcal{B}(\mathbb{R}^n)$. *Moreover*

$$\|T\|_{\mathcal{M}^{1,1}} = \|\mu\|\,.$$

Proof. If $\mu \in \mathcal{B}(\mathbb{R}^n)$, then $Tf = f * \mu$ is a translation invariant operator and

$$\|T\|_{\mathcal{M}^{1,1}} \leq \|\mu\| \tag{4.2}$$

by Theorem 2.5. Suppose now that $T \in \mathcal{M}^{1,1}$. Then there is $u \in \mathcal{S}_n'$ such that

$$T\psi = u * \psi \quad \text{for } \psi \in \mathcal{S}_n.$$

Let $\varphi \in \mathcal{S}_n$, $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. It follows from Problem 14 that for $\psi \in \mathcal{S}_n$, $\tilde{\varphi}_\varepsilon * \psi \to \psi$ in the topology of $\mathcal{S}_n$ as $\varepsilon \to 0$. Since $\|\varphi_\varepsilon\|_1 = 1$, the family $\{\varphi_\varepsilon\}_\varepsilon$ is bounded in L^1, so the family

$$\{u * \varphi_\varepsilon\}_\varepsilon = \{T\varphi_\varepsilon\}_\varepsilon \subset L^1(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$$

is also bounded. According to the Riesz representation theorem (Theorem 2.4), the space $\mathcal{B}(\mathbb{R}^n)$ is dual to the separable Banach space $C_0(\mathbb{R}^n)$, and hence it follows from the separable case of the Banach-Alaoglu theorem[21] that there is a weakly-$*$ convergent subsequence

$$u * \varphi_{\varepsilon_k} \overset{*}{\rightharpoonup} \mu \in \mathcal{B}(\mathbb{R}^n) \quad \text{as } k \to \infty,$$

[21]Theorem 14.6 in *Functional Analysis.*

i.e. for every $g \in C_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (u * \varphi_{\varepsilon_k})(x) g(x)\, dx \to \int_{\mathbb{R}^n} g(x)\, d\mu(x)\,. \tag{4.3}$$

We claim that $u = \mu$. Indeed, for $g = \psi \in \mathcal{S}_n$ (4.3) means that

$$(u * \varphi_{\varepsilon_k})[\psi] = u[\tilde{\varphi}_{\varepsilon_k} * \psi] \to \mu[\psi] \quad \text{as } k \to \infty.$$

Since $\tilde{\varphi}_{\varepsilon_k} * \psi \to \psi$ in $\mathcal{S}_n$, continuity of u gives

$$u[\psi] = \mu[\psi] \quad \text{for } \psi \in \mathcal{S}_n,$$

i.e. $u = \mu$. For $g \in C_0(\mathbb{R}^n)$, (4.3) yields

$$\Big| \int_{\mathbb{R}^n} g\, d\mu \Big| \leq \|g\|_\infty \sup_k \|T\varphi_{\varepsilon_k}\|_1 \leq \|g\|_\infty \|T\|_{\mathcal{M}^{1,1}}\,.$$

Now taking the supremum over $g \in C_0(\mathbb{R}^n)$, $\|g\|_\infty \leq 1$ and applying Theorem 2.4 we obtain

$$\|\mu\| \leq \|T\|_{\mathcal{M}^{1,1}}$$

which together with (4.2) yields $\|T\|_{\mathcal{M}^{1,1}} = \|\mu\|$. □

Theorem 4.8. $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ *if and only if there is a function* $m \in L^\infty(\mathbb{R}^n)$ *such that*[22]

$$Tf = \mathcal{F}^{-1}(m(\mathcal{F}f)) \quad \textit{for } f \in L^2(\mathbb{R}^n).$$

Moreover

$$\|T\|_{\mathcal{M}^{2,2}} = \|m\|_\infty\,.$$

Proof. If $m \in L^\infty$, then if follows from the Plancherel theorem that the operator

$$Tf = \mathcal{F}^{-1}(m(\mathcal{F}f))$$

is bounded on L^2. Indeed,

$$\|\mathcal{F}^{-1}(m(\mathcal{F}f))\|_2 = \|m(\mathcal{F}f)\|_2 \leq \|m\|_\infty \|\mathcal{F}f\|_2 = \|m\|_\infty \|f\|_2\,.$$

It is easy to see that the operator T is translation invariant, so $T \in \mathcal{M}^{2,2}$ and the above estimate yields

$$\|T\|_{\mathcal{M}^{2,2}} \leq \|m\|_\infty\,. \tag{4.4}$$

Now let $T \in \mathcal{M}^{2,2}$ and let $u \in \mathcal{S}_n'$ be such that $T\varphi = u * \varphi$ for $\varphi \in \mathcal{S}_n$.

Let $\varphi_0(x) = e^{-\pi|x|^2}$. Recall that $\hat{\varphi}_0 = \varphi_0$ (Theorem 2.14). We have

$$\varphi_0 \hat{u} = \hat{\varphi}_0 \hat{u} = (u * \varphi_0)\hat{} = (T\varphi_0)\hat{} \in L^2(\mathbb{R}^n)\,.$$

Hence

$$m(x) = \frac{(\varphi_0 \hat{u})(x)}{\varphi_0(x)} \in L^2_{\mathrm{loc}}\,.$$

Note that the multiplication by $1/\varphi_0(x) = e^{\pi|x|^2}$ is *not* allowed in $\mathcal{S}_n'$, because $e^{\pi|x|^2}$ is *not* slowly increasing, so we *cannot* conclude that $m = \hat{u}$ (at least

[22]Compare with the operators defined by the formulas (2.20) and (2.21). According to the theorem the operator $(I - \Delta)^{-N}$ is bounded on L^2, while $(I - \Delta)^N$ is not.

not now). However, $m(x)$ is a well defined function in $L^2_{\rm loc}$ and, *of course*, we want to prove that $m=\hat{u}$, but we have to be very cautious and check that every step in our proof is justifiable.

First we will prove that

$$\hat{u}[\psi]=m[\psi]\quad \text{for } \psi\in C_0^\infty(\mathbb{R}^n). \tag{4.5}$$

Observe that for $\psi\in C_0^\infty(\mathbb{R}^n)$ both sides of this equality are well defined with the right hand side understood as

$$m[\psi]=\int_{\mathbb{R}^n} m(x)\psi(x)\,dx\,.$$

We have

$$\begin{aligned} m[\psi] &= \int_{\mathbb{R}^n}\frac{(\varphi_0\hat{u})(x)}{\varphi_0(x)}\,\psi(x)\,dx=\int_{\mathbb{R}^n}(\varphi_0\hat{u})(x)\,\big(\underbrace{\varphi_0^{-1}(x)\psi(x)}_{C_0^\infty}\big)\,dx \\ &= (\varphi_0\hat{u})[\varphi_0^{-1}\psi]=\hat{u}[\varphi_0\varphi_0^{-1}\psi]=\hat{u}[\psi] \end{aligned}$$

which proves (4.5). We could do this calculation only for $\psi\in C_0^\infty$ and not for $\psi\in\mathcal{S}_n$, because we do not know if $m\psi$ is integrable and also, because $\varphi_0^{-1}\psi$ does not necessarily belong to $\mathcal{S}_n$ for $\psi\in\mathcal{S}_n$.

If $\varphi\in C_0^\infty$, then $\varphi m\in L^2\subset\mathcal{S}_n'$ and (4.5) easily implies that

$$\varphi\hat{u}=\varphi m\quad \text{in } \mathcal{S}_n'.$$

Thus

$$\begin{aligned} \|\varphi m\|_2 &= \|\varphi\hat{u}\|_2=\|\big(u*\check{\varphi}\big)\hat{}\,\|_2 \\ &= \|u*\check{\varphi}\|_2=\|T(\check{\varphi})\|_2 \\ &\leq \|T\|_{\mathcal{M}^{2,2}}\|\check{\varphi}\|_2=\|T\|_{\mathcal{M}^{2,2}}\|\varphi\|_2 \end{aligned}$$

and hence

$$\int_{\mathbb{R}^n}\left(\|T\|^2_{\mathcal{M}^{2,2}}-|m(x)|^2\right)|\varphi(x)|^2\geq 0$$

for all $\varphi\in C_0^\infty(\mathbb{R}^n)$. This, however, easily implies that $m\in L^\infty$ with

$$\|m\|_\infty\leq\|T\|_{\mathcal{M}^{2,2}}\,. \tag{4.6}$$

Since $m\in L^\infty$, equality (4.5) yields

$$\hat{u}=m\quad \text{in } \mathcal{S}_n'.$$

Finally for $\varphi\in\mathcal{S}_n$ we have

$$(T\varphi)\hat{}=(u*\varphi)\hat{}=\hat{\varphi}\hat{u}=m\hat{\varphi}\,,$$

so

$$T\varphi=\mathcal{F}^{-1}(m(\mathcal{F}\varphi))$$

and by density

$$Tf=\mathcal{F}^{-1}(m(\mathcal{F}f))\quad \text{for } f\in L^2.$$

Moreover inequalities (4.4) and (4.6) give

$$\|T\|_{\mathcal{M}^{2,2}} = \|m\|_\infty \,.$$

The proof is complete. □

Theorem 4.9. *If* $T \in \mathcal{M}^{p,p}(\mathbb{R}^n)$, $1 \le p < \infty$, *then there is a bounded function* $m \in L^\infty(\mathbb{R}^n)$ *such that*

$$T\varphi = \mathcal{F}^{-1}(m(\mathcal{F}\varphi)) \quad \textit{for } \varphi \in \mathcal{S}_n.$$

In other words we can identify $\mathcal{M}^{p,p}$ *with a subspace of* $\mathcal{M}^{2,2}$.

Proof. Let $1 < p < \infty$ and $T \in \mathcal{M}^{p,p}$. Then $T \in \mathcal{M}^{p',p'}$ by Theorem 4.6. Since 2 is between p and p' it follows from the interpolation theorem[23] that $T \in \mathcal{M}^{2,2}$. If $p = 1$, then $T\varphi = \varphi * \mu = \mathcal{F}^{-1}(m(\mathcal{F}\varphi))$, where $m = \hat{\mu} \in L^\infty$. □

Definition. Given $1 \le p < \infty$ we define $\mathcal{M}_p(\mathbb{R}^n)$ to be the space of bounded functions $m \in L^\infty(\mathbb{R}^n)$ such that the operator

$$T_m\varphi = \mathcal{F}^{-1}(m(\mathcal{F}\varphi)), \quad \varphi \in \mathcal{S}_n$$

is bounded on L^p. The norm of m in $\mathcal{M}_p(\mathbb{R}^n)$ is defined by

$$\|m\|_{\mathcal{M}_p} = \|T_m\|_{\mathcal{M}^{p,p}} \,.$$

Elements of the space $\mathcal{M}_p(\mathbb{R}^n)$ are called L^p *multipliers* or L^p *Fourier multipliers.* The function m is also called the *symbol* of the operator T_m.

It follows from Theorem 4.8 that

$$\mathcal{M}_2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n), \qquad \|m\|_{\mathcal{M}_2} = \|m\|_\infty$$

and Theorem 4.7 shows that $\mathcal{M}_1(\mathbb{R}^n)$ consists of functions that are Fourier transforms of measures of finite bounded variation.

If $1 < q \le 2$ and $m = \hat{\mu} \in \mathcal{M}_1$, then by Theorem 2.5 and Theorem 4.7 we have

$$\|T_m\varphi\|_q = \|\varphi * \mu\|_q \le \|\mu\|\|f\|_q = \|m\|_{\mathcal{M}_1}\|f\|_q \,,$$

so $\mathcal{M}_1 \subset \mathcal{M}_q$ and $\|m\|_{\mathcal{M}_q} \le \|m\|_{\mathcal{M}_1}$.

If $1 < q \le p \le 2$ and $m \in \mathcal{M}_q$, then according to Theorem 4.6

$$T_m : L^q \to L^q, \quad T_m : L^{q'} \to L^{q'}, \quad \|T_m\|_{\mathcal{M}^{q,q}} = \|T_m\|_{\mathcal{M}^{q',q'}} \,.$$

Since $q \le p \le q'$, the Riesz-Thorin theorem gives

$$T_m : L^p \to L^p, \qquad \|T_m\|_{\mathcal{M}^{p,p}} \le \|T_m\|_{\mathcal{M}^{q,q}} \,.$$

Hence $\mathcal{M}_q \subset \mathcal{M}_p$, $\|m\|_{\mathcal{M}_p} \le \|m\|_{\mathcal{M}_q}$. Thus for $1 \le q \le p \le 2$ we have

$$\mathcal{M}_1 \subset \mathcal{M}_q \subset \mathcal{M}_p \subset \mathcal{M}_2 = L^\infty \,,$$

$$\|m\|_\infty \le \|m\|_{\mathcal{M}_p} \le \|m\|_{\mathcal{M}_q} \le \|m\|_{\mathcal{M}_1} \,. \tag{4.7}$$

[23]Riesz-Thorin or Marcinkiewicz.

Finally, if $1 < p \leq 2$, then Theorem 2.7 implies that $\mathcal{M}_p = \mathcal{M}_{p'}$ isometrically.

Example. The function $m(\xi) = e^{2\pi i \xi \cdot h}$ is an L^p multiplicator for all $h \in \mathbb{R}^n$ and the corresponding operator is $T_m f(x) = f(x+h)$.

Theorem 4.10. *If $1 \leq p < \infty$, then $\mathcal{M}_p(\mathbb{R}^n)$ is a commutative Banach algebra with respect to a pointwise multiplication.*

Proof. Since $T_{am_1+bm_2} = aT_{m_1} + bT_{m_2}$ it follows that $\mathcal{M}_p$ is a linear space and

$$\|am_1 + bm_2\|_{\mathcal{M}_p} = \|T_{am_1+bm_2}\|_{\mathcal{M}^{p,p}} \leq |a|\|m_1\|_{\mathcal{M}^p} + |b|\|m_2\|_{\mathcal{M}^p}$$

shows that $\|\cdot\|_{\mathcal{M}_p}$ is a norm. Since $T_{m_1 m_2} = T_{m_1} T_{m_2}$ we have

$$\|m_1 m_2\|_{\mathcal{M}_p} = \|T_{m_1 m_2}\|_{\mathcal{M}^{p,p}} \leq \|m_1\|_{\mathcal{M}_p} \|m_2\|_{\mathcal{M}_p}\,,$$

so $\mathcal{M}_p$ is a complex commutative algebra with the unit element $m \equiv 1$, $\|m\|_{\mathcal{M}_p} = 1$ that corresponds to the identity mapping.

Thus it remains to prove that $\mathcal{M}_p$ is complete with respect to the norm. If $2 < p < \infty$, then $\mathcal{M}_p$ is isometric to $\mathcal{M}_{p'}$, $1 < p' < 2$, so we can assume that $1 \leq p \leq 2$. Let $\{m_k\}$ be a Cauchy sequence in $\mathcal{M}_p$. By (4.7)

$$\|m_j - m_l\|_\infty \leq \|m_j - m_k\|_{\mathcal{M}_p}$$

and hence $\{m_j\}$ is a Cauchy sequence in L^∞. Thus it converges in the L^∞ norm to a bounded function $m \in L^\infty$. We have to prove that $m \in \mathcal{M}_p$ and $m_j \to m$ in $\mathcal{M}_p$.

Fix $\varphi \in \mathcal{S}_n$. The dominated convergence theorem yields

$$(T_{m_k}\varphi)(x) = \int_{\mathbb{R}^n} \hat{\varphi}(\xi) m_k(\xi) e^{2\pi i x\cdot\xi}\, d\xi \to \int_{\mathbb{R}^n} \hat{\varphi}(\xi) m(\xi) e^{2\pi i x\cdot\xi}\, d\xi = (T_m\varphi)(x)\,.$$

Given $\varepsilon > 0$ let N be such that $\|T_{m_j} - T_{m_k}\|_{\mathcal{M}^{p,p}} < \varepsilon$ for $j, k \geq N$. For $j \geq N$ Fatou's lemma implies

$$\begin{aligned}
\int_{\mathbb{R}^n} |(T_{m_j} - T_m)\varphi|^p\, dx &= \int_{\mathbb{R}^n} \lim_{k\to\infty} |(T_{m_j} - T_{m_k})\varphi|^p\, dx \\
&\leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} |(T_{m_j} - T_{m_k})\varphi|^p\, dx \\
&\leq \varepsilon^p \|\varphi\|_p^p\,,
\end{aligned}$$

i.e. $\|T_{m_j} - T_m\|_{\mathcal{M}^{p,p}} \leq \varepsilon$. Thus $T_m \in \mathcal{M}^{p,p}$ and $T_{m_j} \to T_m$ in $\mathcal{M}^{p,p}$, i.e. $m \in \mathcal{M}_p$ and $m_j \to m$ in $\mathcal{M}_p$. □

The following result easily follows from the properties of the Fourier transform.

Theorem 4.11. *Let $m \in \mathcal{M}_p(\mathbb{R}^n)$, $1 \leq p < \infty$, $x \in \mathbb{R}^n$ and $h > 0$. Then*

(a) $\|\tau_x m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}$.

(b) $\|\delta^h m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}$, *where* $(\delta^h m)(x) = m(hx)$ *is a dilation.*
(c) $\|\tilde{m}\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}$.
(d) $\|e^{2\pi i x\cdot(\cdot)} m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}$.
(e) $\|m(\rho\cdot)\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}$, *when* $\rho \in O(n)$ *is an orthogonal transformation.*

The following result is often useful.

Proposition 4.12. *Let* $m \in \mathcal{M}_p(\mathbb{R}^n)$, $1 \le p < \infty$ *and* $\psi \in L^1(\mathbb{R}^n)$. *Then* $m * \psi \in \mathcal{M}_p(\mathbb{R}^n)$ *and*

$$\|m * \psi\|_{\mathcal{M}_p} \le \|\psi\|_1 \|m\|_{\mathcal{M}_p} .$$

We leave the proof as an exercise.

In particular if $\psi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \psi \, dx = 1$ and $\psi_\varepsilon = \varepsilon^{-n}\psi(x/\varepsilon)$, then $m_\varepsilon = m * \psi_\varepsilon$ satisfies $m_\varepsilon \in C^\infty$, $m_\varepsilon \in \mathcal{M}_p$, $\|m_\varepsilon\|_{\mathcal{M}_p} \le \|m\|_{\mathcal{M}_p}$. Hence for every $\varphi \in \mathcal{S}_n$

$$(T_{m_\varepsilon}\varphi)(x) \to (T_m \varphi)(x) \quad \text{as } \varepsilon \to 0$$

for every $x \in \mathbb{R}^n$.

The next result is a kind of Fubini theorem for Fourier multipliers.

Theorem 4.13. *Suppose that* $m(\xi, \eta) \in \mathcal{M}_p(\mathbb{R}^{n+m})$, $1 < p < \infty$. *Then for almost every* $\xi \in \mathbb{R}^n$, *the function* $\eta \mapsto m(\xi, \eta)$ *is in* $\mathcal{M}_p(\mathbb{R}^n)$ *and*

$$\|m(\xi, \cdot)\|_{\mathcal{M}_p(\mathbb{R}^m)} \le \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})} .$$

Proof. Since $m \in L^\infty(\mathbb{R}^{n+m})$ it follows from the Fubini theorem that for a.e. $\xi \in \mathbb{R}^n$, $m(\xi, \cdot) \in L^\infty(\mathbb{R}^m)$ and

$$\|m(\xi, \cdot)\|_\infty \le \|m\|_\infty . \tag{4.8}$$

Fix $\varphi_1, \psi_1 \in \mathcal{S}_n$ and $\varphi_2, \psi_2 \in \mathcal{S}_m$. For ξ such that (4.8) is satisfied we define

$$M(\xi) = \int_{\mathbb{R}^m} (m(\xi, \cdot)\hat{\varphi}_2(\cdot))^\vee(\eta)\psi_2(\eta) \, d\eta = \int_{\mathbb{R}^m} m(\xi, \eta)\hat{\varphi}_2(\eta)\check{\psi}_2(\eta) \, d\eta .$$

Observe that $M \in L^\infty(\mathbb{R}^n)$ with

$$\|M\|_\infty \le \|m\|_\infty \|\hat{\varphi}_2 \check{\psi}_1\|_1 .$$

Thus M defines a multiplier (at least an L^2 multiplier) and we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}(M(\cdot)\hat{\varphi}_1(\cdot))^\vee(\xi)\psi_1(\xi)\,d\xi\Big| &= \Big|\int_{\mathbb{R}^n} M(\xi)\hat{\varphi}_1(\xi)\check{\psi}_1(\xi)\,d\xi\Big| \\
= & \Big|\iint_{\mathbb{R}^{n+m}} m(\xi,\eta)\hat{\varphi}_1(\xi)\hat{\varphi}_2(\eta)\check{\psi}_1(\xi)\check{\psi}_2(\eta)\,d\eta\,d\xi\Big| \\
= & \Big|\iint_{\mathbb{R}^{n+m}} m(\xi,\eta)(\varphi_1\varphi_2)^\wedge(\xi,\eta)(\psi_1\psi_2)^\vee(\xi,\eta)\,d\xi\,d\eta\Big| \\
= & \Big|\iint_{\mathbb{R}^{n+m}} \big(m(\varphi_1\varphi_2)^\wedge\big)^\vee(\psi_1\psi_2)\,d\xi\,d\eta\Big| \\
\leq & \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})}\|\varphi_1\|_p\|\varphi_2\|_p\|\psi_1\|_{p'}\|\psi_2\|_{p'}\,.
\end{aligned}$$

Taking the supremum over $\psi_1\in\mathcal{S}_n$ with $\|\psi_1\|_{p'}\leq 1$ it follows that

$$\big\|(M\hat{\varphi}_1)^\vee\big\|_p \leq \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})}\|\varphi_2\|_p\|\psi_2\|_{p'}\|\varphi_1\|_p\,,$$

i.e. $M\in\mathcal{M}_p(\mathbb{R}^n)$ with

$$\|M\|_{\mathcal{M}_p(\mathbb{R}^n)} \leq \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})}\|\varphi_2\|_p\|\psi_2\|_{p'}\,.$$

Since $\|M\|_\infty\leq\|M\|_{\mathcal{M}_p(\mathbb{R}^n)}$ we conclude that

$$\Big|\int_{\mathbb{R}^m}\big(m(\xi,\cdot)\hat{\varphi}_2(\cdot)\big)^\vee(\eta)\psi_2(\eta)\,d\eta\Big| \leq \|m\|_{\mathcal{M}_p(\mathbb{R}^{n+m})}\|\varphi_2\|_p\|\psi_2\|_{p'}$$

and taking the supremum over $\|\psi_2\|_{p'}\leq 1$ yields the result. □

The above theory of Fourier multipliers is very beautiful, but there is one problem: we do not have good examples. Indeed, we characterized all L^2 multipliers as bounded functions, but we do not know what bounded functions define L^p multipliers for $p\neq 2$. The only examples of L^p multipliers that we know so far come from translation invariant operators on L^1. Indeed, every such an operator is a convolution with a measure and it is also bounded in L^p for all p. Thus all functions that are Fourier transforms of measures of finite total variation define L^p multipliers, but on the other hand there is no need to use the theory of Fourier multipliers to deal with convolutions of measures and there is a huge gap between the space of Fourier transforms of measures and all bounded functions. In the following sections we will construct more and more L^p multipliers, but as we shall see it is always a very difficult task.

5. The Hilbert transform

The function $f(x)=\sin x/x$ is not integrable on $[0,\infty)$, but we define its integral as an improper one

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{R\to\infty}\int_0^R\frac{\sin x}{x}\,dx = \frac{\pi}{2}\,. \tag{5.1}$$

A similar problem appears if we want to define the integral

$$\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

Since the function $1/x$ is not integrable in any neighborhood of 0, in the case in which $\varphi(0) \neq 0$, the integral diverges. This suggests that we should define this integral as a kind of an improper integral known as the *principal value of the integral*

$$\left(\text{p.v.}\, \frac{1}{x}\right)[\varphi] = \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx. \tag{5.2}$$

The limit exists when $\varphi \in \mathcal{S}(\mathbb{R})$ and it actually defines a tempered distribution.

Theorem 5.1. *If $\varphi \in \mathcal{S}(\mathbb{R})$, then the limit at (5.2) exists and defines a tempered distribution* p.v. $1/x \in \mathcal{S}'(\mathbb{R})$ *such that*

$$\left| \left(\text{p.v.}\, \frac{1}{x}\right)[\varphi] \right| \leq 2(\|\varphi'\|_\infty + \|x\varphi\|_\infty).$$

Proof. Note that

$$\int_{\varepsilon \leq |x| \leq 1} \frac{\varphi(0)}{x}\, dx = 0,$$

so we have

$$\int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx = \int_{\varepsilon \leq |x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x}\, dx + \int_{|x| \geq 1} \frac{\varphi(x)}{x}\, dx.$$

For the first integral on the right hand side we have

$$\begin{aligned} \int_{\varepsilon \leq |x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x}\, dx &= \int_{\varepsilon \leq |x| \leq 1} \int_0^1 \varphi'(tx)\, dt\, dx \\ &\to \int_{|x| \leq 1} \int_0^1 \varphi'(tx)\, dt\, dx \quad \text{as } \varepsilon \to 0 \end{aligned}$$

and hence

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx = \int_{|x| \leq 1} \int_0^1 \varphi'(tx)\, dt\, dx + \int_{|x| \geq 1} \frac{\varphi(x)}{x}\, dx.$$

Since

$$\left| \int_{|x| \geq 1} \frac{\varphi(x)}{x}\, dx \right| \leq \sup_{x \in \mathbb{R}} |x\varphi(x)| \int_{|x| \geq 1} \frac{dx}{x^2} = 2\|x\varphi\|_\infty$$

and

$$\left| \int_{|x| \leq 1} \int_0^1 \varphi'(tx)\, dt\, dx \right| \leq 2\|\varphi'\|_\infty$$

the theorem follows. □

Exercise. *Prove that for* $\varphi \in \mathcal{S}(\mathbb{R})$

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \varphi'(x) \left| \ln\left(\frac{|x|}{R}\right)\right| dx\,.$$

Since p.v. $1/x$ is a tempered distribution, we may try to compute its Fourier transform.

Theorem 5.2.

$$\left(\frac{1}{\pi}\,\text{p.v.}\,\frac{1}{x}\right)^{\wedge}(\xi) = -i\,\text{sgn}\,(\xi)\,.$$

Proof. For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\left(\text{p.v.}\,\frac{1}{x}\right)[\varphi] = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\, dx\,,$$

so

$$g_\varepsilon(x) = \frac{1}{x}\,\chi_{\{|x|\geq\varepsilon\}} \to \text{p.v.}\,\frac{1}{x} \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

and hence

$$\hat{g}_\varepsilon \to \left(\text{p.v.}\,\frac{1}{x}\right)^{\wedge} \quad \text{in } \mathcal{S}'(\mathbb{R}) \text{ as } \varepsilon \to 0. \tag{5.3}$$

Since $g_\varepsilon \in L^2$ we can compute its Fourier transform using Theorem 2.31.[24]

$$\begin{aligned}
\hat{g}_\varepsilon(\xi) &= \lim_{R\to\infty} \int_{-R}^{R} g_\varepsilon(x) e^{-2\pi i x\xi}\, dx \\
&= \lim_{R\to\infty} \int_{\varepsilon}^{R} \left(\frac{1}{x}\, e^{-2\pi i x\xi} + \frac{1}{(-x)}\, e^{-2\pi i(-x)\xi}\right) dx \\
&= \lim_{R\to\infty} \int_{\varepsilon}^{R} \frac{e^{-2\pi i x\xi} - e^{2\pi i x\xi}}{x}\, dx \\
&= \lim_{R\to\infty} -2i \int_{\varepsilon}^{R} \frac{\sin(2\pi x\xi)}{x}\, dx \\
&= \lim_{R\to\infty} -2i\,\text{sgn}\,(\xi) \int_{\varepsilon}^{R} \frac{\sin(2\pi x|\xi|)}{x}\, dx \\
&= \lim_{R\to\infty} -2i\,\text{sgn}\,(\xi) \int_{2\pi|\xi|\varepsilon}^{2\pi|\xi|R} \frac{\sin y}{y}\, dy \\
&= -2i\,\text{sgn}\,(\xi) \int_{2\pi|\xi|\varepsilon}^{\infty} \frac{\sin y}{y}\, dy\,,
\end{aligned}$$

[24]The limit is understood in the L^2 sense, but we will prove that the limit also exists in the pointwise sense, so the pointwise limit must be equal to the L^2 one.

where the limit exists as an improper integral, see (5.1). Note that the above computation gives also

$$|\hat{g}_\varepsilon(\xi)| \le M \quad \text{independntly of } \varepsilon \tag{5.4}$$

and

$$\lim_{\varepsilon\to 0} \hat{g}_\varepsilon(\xi) = -2i \operatorname{sgn}(\xi)\,\frac{\pi}{2} = -\pi i \operatorname{sgn}(\xi)\,. \tag{5.5}$$

Thus (5.3) yields the result. □

DEFINITION. The *Hilbert transform* of a function f is defined by

$$\begin{aligned} Hf(x) &= \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\,dy \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon} \frac{f(x-y)}{y}\,dy \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon} \frac{f(y)}{x-y}\,dy \end{aligned}$$

and the question is for what functions and in what sense the limit exists.

Note that if $\varphi \in \mathcal{S}(\mathbb{R})$ and $u = \text{p.v.}\,1/x \in \mathcal{S}'(\mathbb{R})$, then

$$H\varphi(x) = \frac{1}{\pi}u[\varphi(x-\cdot)] = \frac{1}{\pi}(u*\varphi)(x)\,,$$

so $H\varphi \in C^\infty$, and $H\varphi$ and all its derivatives are slowly increasing (see Theorem 2.36). Note also that

$$(H\varphi)^\wedge(\xi) = \frac{1}{\pi}(\hat{u}\hat{\varphi})(\xi) = -i\operatorname{sgn}(\xi)\hat{\varphi}(\xi)$$

and hence

$$H\varphi(x) = \big(-i\operatorname{sgn}(\cdot)\hat{\varphi}(\cdot)\big)^\vee(x)\,.$$

Since $m(\xi) = -i\operatorname{sgn}(\xi) \in L^\infty(\mathbb{R})$, the Hilbert transform defines a translation invariant operator on L^2. Actually we have.

Theorem 5.3. *If $f \in L^2(\mathbb{R})$, then*

$$Hf(x) = \big(-i\operatorname{sgn}(\cdot)\hat{f}(\cdot)\big)^\vee(x)\,. \tag{5.6}$$

Hence $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry of L^2 onto L^2,

$$\|Hf\|_2 = \|f\|_2 \quad \textit{for } f \in L^2(\mathbb{R})$$

with the inverse operator satisfying

$$H^{-1} = -H\,.$$

Moreover if

$$H^\varepsilon f = \frac{1}{\pi}\int_{|x-y|\ge\varepsilon} \frac{f(y)}{x-y}\,dy\,,$$

then

$$H^\varepsilon f \to Hf \quad \textit{in } L^2 \textit{ as } \varepsilon \to 0.$$

Proof. We proved (5.6) for $f = \varphi \in \mathcal{S}(\mathbb{R})$. Since $m(\xi) = -i\,\mathrm{sgn}\,(\xi)$ is bounded, H uniquely extends to a translation invariant operator $H \in \mathcal{M}^{2,2}$ and (5.6) follows from the Plancherel theorem. Another application of the Plancherel theorem gives

$$\|Hf\|_2 = \|\widehat{Hf}\|_2 = \| - i\,\mathrm{sgn}\,(\cdot)\hat{f}(\cdot)\|_2 = \|\hat{f}\|_2 = \|f\|_2\,.$$

Moreover

$$H^2 f = \big((-i\,\mathrm{sgn}\,(\xi))^2\hat{f}(\xi)\big)^{\vee} = -f\,,$$

so $H^2 = -I$ and hence $H^{-1} = -H$. Finally if

$$H^{\varepsilon} f(x) = \frac{1}{\pi}\int_{|x-y|\geq\varepsilon}\frac{f(y)}{x-y}\,dy = \frac{1}{\pi}\big(g_\varepsilon * f\big)(x)\,,$$

where

$$g_\varepsilon(x) = \frac{1}{x}\,\chi_{\{|x|\geq\varepsilon\}} \in L^2$$

formulas (5.4) and (5.5) give

$$|\hat{g}_\varepsilon(\xi)| \leq M \quad \text{independently of } \varepsilon$$

and

$$\hat{g}_\varepsilon(\xi) \to -\pi i\,\mathrm{sgn}\,(\xi) \quad \text{as } \varepsilon\to 0 \text{ for all } \xi\in\mathbb{R}.$$

Hence

$$\big(H^\varepsilon f\big)^{\wedge}(\xi) = \frac{1}{\pi}\hat{f}(\xi)\hat{g}_\varepsilon(\xi) \to -i\,\mathrm{sgn}\,(\xi)\hat{f}(\xi)$$

as $\varepsilon \to 0$ and

$$\big|(H^\varepsilon f)^{\wedge}(\xi)\big| \leq \frac{M}{\pi}|\hat{f}(\xi)|\,.$$

Thus the dominated convergence theorem yields

$$\big(H^\varepsilon f\big)^{\wedge} \to -i\,\mathrm{sgn}\,(\xi)\hat{f}(\xi) \quad \text{in } L^2 \text{ as } \varepsilon\to 0$$

and hence

$$H^\varepsilon f \to \big(-i\,\mathrm{sgn}\,(\xi)\hat{f}(\xi)\big)^{\vee} = Hf \quad \text{in } L^2$$

as $\varepsilon \to 0$ by Plancherel's theorem. □

5.1. **Conjugate harmonic functions.** We will see now that the Hilbert transform arises naturally in connection with boundary behavior of holomorphic functions.

Let us recall that the Poisson kernel

$$P_t(x) = P(x,t) = c_n\,\frac{t}{(t^2+|x|^2)^{(n+1)/2}}\,, \quad x\in\mathbb{R}^n,\ t>0\,.$$

The function $P(x,t)$ is harmonic in $\mathbb{R}^{n+1}_+$. One can check it by a direct computation, but it is worth to note that this also follows the fact that for $n \geq 2$

$$P(x,t) = -\frac{c_n}{n-1}\frac{\partial}{\partial t}\left(\frac{1}{(t^2+|x|^2)^{(n-1)/2}}\right)$$

and

$$\frac{1}{(t^2+|x|^2)^{(n-1)/2}} = \|(t,x)\|^{2-(n+1)}$$

is the standard radial harmonic function in $\mathbb{R}^{n+1}\setminus\{0\}$.[25] Similar argument works also for $n=1$.

If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then the Poisson integral

$$u(x,t) = \int_{\mathbb{R}^n} P(x-y,t)f(y)\,dy = P_t * f(x)$$

is harmonic in $\mathbb{R}^{n+1}_+$. Indeed, it is easy to see that we can differentiate u under the sign of the integral and hence harmonicity of u follows from that of $P(x,t)$.

Note that (2.7) and Theorem 2.28 imply that

$$u(\cdot,t) \to f(\cdot) \tag{5.7}$$

both in $L^p(\mathbb{R}^n)$ and a.e. as $t \to 0^+$. Thus the convolution with the Poisson kernel solves the following Dirichlet problem.

$$\begin{cases} \Delta_{(x,t)}u(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \\ u(x,0) = f(x), \end{cases}$$

where the boundary condition $u(x,0)=f(x)$ is understood in the sense of the limit (5.7).

Now let us restrict to the case $n=1$, so

$$P_y(x) = P(x,y) = \frac{1}{\pi}\,\frac{y}{y^2+x^2}\,, \quad x\in\mathbb{R},\ y>0\,,$$

where for convenience reasons we use variable y instead of t. Thus for a real valued function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$,

$$u(x,y) = (f*P_y)(x) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+y^2}\,dt$$

is a solution to a corresponding Dirichlet problem in the upper half-plane $\mathbb{R}^2_+ = \{(x,y):\ y>0\}$.

Every harmonic function in $\mathbb{R}^2_+$ is a real part of a holomorphic one. Clearly the function

$$F(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}\,dt$$

is holomorphic in $\mathbb{R}^2_+ = \{\operatorname{im} z > 0\}$ and[26]

$$\operatorname{re} F(z) = u(x,y)\,.$$

[25]Up to a constant it is the fundamental solution to the Laplace operator Δ.

[26]Since $\operatorname{re}(i/(z-t)) = y/((x-t)^2+y^2)$.

Also

$$\operatorname{im} F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-t)f(t)}{(x-t)^2 + y^2}\, dt = (f * Q_y)(x)\,,$$

where

$$Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2}$$

is called the *conjugate Poisson kernel*.

The functions

$$u(x+iy) = (f * P_y)(x), \qquad v(x+iy) = (f * Q_y)(x)$$

are conjugate harmonic functions in $\mathbb{R}^2_+$ since they are real and imaginary parts of the holomorphic function $F(z)$. We know that

$$u(\cdot, y) \to f(\cdot) \quad \text{as } y \to 0^+$$

both in L^p and a.e. and it is natural to ask what is the limit of $v(x, y) = (f * Q_y)(x)$ as $y \to 0^+$?

Formally

$$\lim_{y\to 0^+} Q_y(x) = \frac{1}{\pi x}$$

so it should not be surprising that the limit of $f * Q_y$ as $y \to 0^+$ equals to the Hilbert transform Hf.

Theorem 5.4. *For $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ we have*

$$f * Q_\varepsilon - H^\varepsilon f \to 0 \quad \text{as } \varepsilon \to 0^+$$

both in $L^p(\mathbb{R})$ and a.e. More precisely $f * Q_\varepsilon(x) - H^\varepsilon f(x) \to 0$ whenever x is a Lebesgue point of f.

Proof. Note that

$$f * Q_\varepsilon(x) - H^\varepsilon f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_\varepsilon(x-t) f(t)\, dt = \frac{1}{\pi}\, (f * \psi_\varepsilon)(x)\,,$$

where

$$\psi_\varepsilon(x) = \frac{x}{x^2 + \varepsilon^2} - \frac{1}{x} \chi_{\{|x| \geq \varepsilon\}}\,.$$

Note also that if

$$\psi(x) = \psi_1(x) = \frac{x}{x^2 + 1} - \frac{1}{x} \chi_{\{|x| \geq 1\}}\,.$$

then $\psi_\varepsilon(x) = \varepsilon^{-1} \psi(x/\varepsilon)$. The function

$$\Psi(x) = \frac{1}{x^2 + 1}$$

is an integrable radially decreasing majorant of ψ, i.e. $|\psi| \leq |\Psi|$ and $\int_{\mathbb{R}} \psi = 0$. Hence Theorem 2.28 with $a = 0$ implies that

$$f * Q_\varepsilon - H^\varepsilon f \to 0 \quad \text{a.e.}$$

The convergence to 0 in L^p follows from Corollary 2.12. □

If $\varphi \in \mathcal{S}(\mathbb{R})$, then $H^\varepsilon\varphi \to H\varphi$ everywhere by Theorem 5.1. Since every point of φ is Lebesgue, Theorem 5.4 gives

Corollary 5.5. *If $\varphi \in \mathcal{S}(\mathbb{R})$, then*

$$\varphi * Q_\varepsilon(x) \to H\varphi(x) \quad \textit{as } \varepsilon \to 0^+$$

for every $x \in \mathbb{R}$.

Since for $f \in L^2$, $H^\varepsilon f \to Hf$ in L^2 we immediately get

Corollary 5.6. *If $f \in L^2(\mathbb{R})$, then*

$$f * Q_\varepsilon \to Hf \quad \textit{in } L^2 \textit{ as } \varepsilon \to 0^+.$$

Later we will see that $H^\varepsilon f \to Hf$ both in L^p and a.e. for any $f \in L^p$, $1 < p < \infty$ and hence the corollary extends to $1 < p < \infty$. This will show that if $f \in L^p(\mathbb{R})$, $1 < p < \infty$ is the boundary value of the real part of the holomorphic function F in $\mathbb{R}^2_+$, then the boundary value of the imaginary part is $Hf \in L^p$.

5.2. L^p **estimates.** The boundedness of the Hilbert transform in L^p, $1 < p < \infty$ plays a fundamental role in harmonic analysis and its applications. In this section we will show two different proofs of this result and in one of the following sections we will provide one more proof. The Hilbert transform is the simplest example of a singular integral and the result is a special case of L^p estimates for singular integrals that will be discussed in later sections.

Theorem 5.7 (Riesz)**.** *If $1 < p < \infty$, then there is a constant $C(p) > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$*

$$\|H\varphi\|_p \le C(p)\|\varphi\|_p\,.$$

Moreover $C(p) = C(p')$ and $C(2) = 1$. Thus $H \in \mathcal{M}^{p,p}$ for all $1 < p < \infty$ and hence $m(\xi) = \operatorname{sgn}(\xi) \in \mathcal{M}_p(\mathbb{R})$.

Proof. We proved the result for $p = 2$ with $C(2) = 1$ in Theorem 5.3. Since $T \in \mathcal{M}^{p,p}$ if and only if $T \in \mathcal{M}^{p',p'}$ with $\|T\|_{\mathcal{M}^{p,p}} = \|T\|_{\mathcal{M}^{p',p'}}$ (Theorem 4.6) is suffices to prove the result for $2 < p < \infty$.

We will need the following lemmas which are of independent interest.

Lemma 5.8. *If $u \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$, then $|u|^p \in C^1(\Omega)$ and $\nabla|u|^p = p|u|^{p-2}u\nabla u$. In particular $\nabla|u|^p(x) = 0$ if $u(x) = 0$.*

Proof. Clearly $|u|^p$ is C^1 on the open set where $u \ne 0$ and the formula for $\nabla|u|^p$ is easy to verify on that set. Thus it remains to prove that $|u|^p$ is differentiable on the set where $u = 0$ with $\nabla|u|^p = 0$ on that set. We leave details as an exercise. □

Now if $p > 2$ and $u \in C^2(\Omega)$, then $|u|^p \in C^2(\Omega)$. Indeed,

$$\frac{\partial}{\partial x_i}|u|^p = p|u|^{p-2}u\frac{\partial u}{\partial x_i} = p\,\mathrm{sgn}\,(u)|u|^{p-1}\frac{\partial u}{\partial x_i}\,.$$

The lemma gives $|u|^{p-1}\partial u/\partial x_i \in C^1$ and

$$\frac{\partial}{\partial x_j}\left(|u|^{p-1}\frac{\partial u}{\partial x_i}\right) = 0 \quad \text{if } u(x) = 0\,,$$

so it is easy to see that

$$\frac{\partial^2}{\partial x_j}\partial x_i\,|u|^p = \frac{\partial}{\partial x_j}\left(p\,\mathrm{sgn}\,(u)|u|^{p-1}\frac{\partial u}{\partial x_i}\right) = p\,\mathrm{sgn}\,(u)\frac{\partial}{\partial x_j}\left(|u|^{p-1}\frac{\partial u}{\partial x_i}\right)$$

and hence the second derivatives of $|u|^p$ are continuous. In particular if $u \in C^2(\Omega)$ and $p > 2$ we can compute $\Delta|u|^p$. Easy calculations give

Lemma 5.9. *If $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ and $p > 2$, then $|u|^2 \in C^2(\Omega)$ and*

$$\Delta|u|^p = p|u|^{p-2}\big(u\Delta u + (p-1)|\nabla u|^2\big)\,.$$

Using a variant of the above argument combined with the Cauchy-Riemann equations one can easily prove

Lemma 5.10. *If $F(z) = u + iv$ is holomorphic in $\Omega \subset \mathbb{C}$ and $p > 2$, then $|F|^p \in C^2(\Omega)$ and*

$$\Delta|F|^p = p^2|F|^{p-2}(u_x^2 + u_y^2)\,.$$

Now we can return to the proof of the theorem. By density of $C_0^\infty(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$ it suffices to assume that $\varphi \in C_0^\infty(\mathbb{R})$. This assumption will simplify some estimates. Let

$$F(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(t)}{z-t}\,dt\,.$$

Then as we have seen in Section 5.1

$$\mathrm{re}\,F(z) = u(x,y) = (\varphi * P_y)(x) \quad \mathrm{im}\,F(z) = v(x,y) = (\varphi * Q_y)(x)\,,$$

where

$$P_y(x) = \frac{1}{\pi}\,\frac{y}{y^2+x^2}, \quad Q_y(x) = \frac{1}{\pi}\,\frac{x}{x^2+y^2}\,.$$

Moreover

$$v(x,y) = (\varphi * Q_y)(x) \to H\varphi(x) \quad \text{as } y \to 0^+$$

for every $x \in \mathbb{R}$, see Corollary 5.5.

Since v is harmonic Lemma 5.9 gives

$$\Delta|v|^p = p(p-1)|v|^{p-2}(v_x^2 + v_y^2) = p(p-1)|v|^{p-2}(u_x^2 + y_y^2)\,,$$

where in the last equality follows from the Cauchy-Riemann equations. This and Lemma 5.10 imply

$$\Delta\left(|F|^p - \frac{p}{p-1}\,|v|^p\right) = p^2(|F|^{p-2} - |v|^{p-2})(u_x^2 + u_y^2) \geq 0\,. \tag{5.8}$$

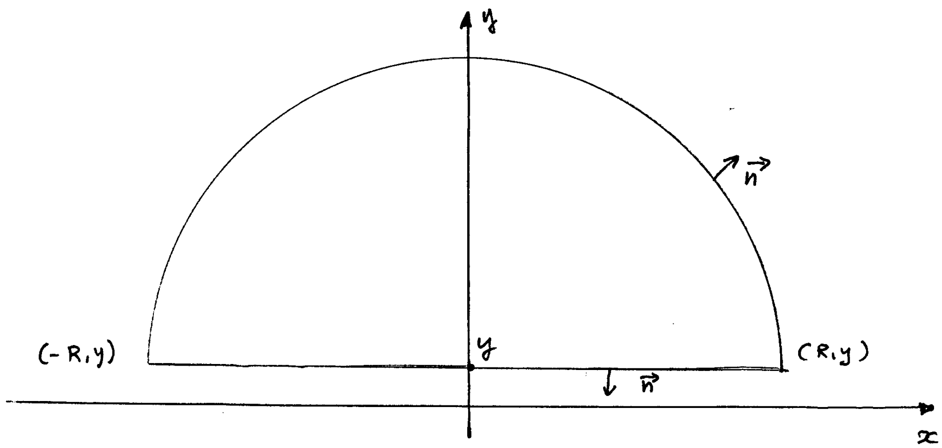
Let us recall that if $f \in C^1(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary and $\vec{n}$ is the outer normal vector to the boundary, then

$$\int_{\partial\Omega} \frac{\partial f}{\partial \vec{n}}\, ds = \int_{\partial\Omega} \nabla f \cdot \vec{n}\, ds = \iint_\Omega \Delta f\, dx\, dy\,. \tag{5.9}$$

This formula is known as Green's identity. We want to apply it to the function

$$f = |F|^p - \frac{p}{p-1}\,|v|^p$$

and Ω as on the picture.



The integral of $\nabla f \cdot \vec{n}$ along the boundary is nonnegative by (5.9) and (5.8). Elementary but tedious estimates[27] show that the part of the integral corresponding to the semi-circle converges to 0 as $R \to \infty$. Thus for every $y > 0$ we obtain[28]

$$\frac{\partial}{\partial y}\, I(y) = \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \left(|F(x,y)|^p - \frac{p}{p-1}\,|v(x,y)|^p \right) dx \le 0 \tag{5.10}$$

Hence the function

$$I(y) = \int_{-\infty}^{\infty} \left(|F(x,y)|^p - \frac{p}{p-1}\,|v(x,y)|^p \right) dx$$

is decreasing. The same estimates imply also that $I(y) \to 0$ as $y \to \infty$, so $I(y) \ge 0$ for all $y > 0$ and thus

$$\int_{-\infty}^{\infty} |F(x,y)|^p\, dx \ge \frac{p}{p-1} \int_{-\infty}^{\infty} |v(x,y)|^p\, dx\,.$$

---

[27]Since φ has compact support, the integral formulas that define u and v show that the growth of $u, v, \nabla u, \nabla v$ can be estimated by $C/(x^2+y^2)^{1/2}$ and hence $|f| \le C/(x^2+y^2)^{p/2}$, $|\nabla f| \le C/(x^2+y^2)^{p/2}$.

[28]Note that $\partial/\partial y = -\partial/\partial \vec{n}$ along this line.

Observe that

$$\left(\int_{-\infty}^{\infty}|F(x,y)|^p\,dx\right)^{2/p}=\|u^2(\cdot,y)+v^2(\cdot,y)\|_{p/2}\leq\|u^2(\cdot,y)\|_{p/2}+\|v^2(\cdot,y)\|_{p/2}\,.$$

Hence

$$\left(\frac{p}{p-1}\right)^{2/p}\|v^2(\cdot,y)\|_{p/2}\leq\|u^2(\cdot,y)\|_{p/2}+\|v^2(\cdot,y)\|_{p/2}\,,$$

$$\|v^2(\cdot,y)\|_{p/2}\leq\frac{1}{\left(\frac{p}{p-1}\right)^{2/p}-1}\|u^2(\cdot,y)\|_{p/2}\,. \tag{5.11}$$

Note that $u(\cdot,y)=\varphi*P_y\to\varphi$ in L^p as $y\to 0^+$ by (2.7) and $v(x,y)\to H\varphi(x)$ for every $x\in\mathbb{R}$. Thus letting $y\to 0^+$ in (5.11) and applying Fatou's lemma we obtain

$$\int_{-\infty}^{\infty}|H\varphi(x)|^p\,dx\leq\frac{1}{\left(\left(\frac{p}{p-1}\right)^{2/p}-1\right)^{p/2}}\int_{-\infty}^{\infty}|\varphi(x)|^p\,dx\,.$$

The proof is complete. □

Now we will present another proof of Theorem 5.7 based on the following interesting identity.

Lemma 5.11 (Cotlar)**.** *If $\varphi\in\mathcal{S}(\mathbb{R})$ is a real valued function, then*

$$H(\varphi)^2=\varphi^2+2H(\varphi H(\varphi))\,.$$

Proof. Before we give a rigorous proof we will show a heuristic argument that leads to this identity. As we have already seen, the function $\varphi+iH(\varphi)$ has a holomorphic extension to the upper half-plane. Namely

$$F(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(t)}{z-t}\,dt\,.$$

Hence also the function

$$(\varphi+iH(\varphi))^2=\varphi^2-H(\varphi)^2+i2\varphi H(\varphi)$$

has a holomorphic extension $F(z)^2$. Thus we may expect that[29]

$$2\varphi H(\varphi)=H(\varphi^2-H(\varphi)^2)$$

Applying H to both sides and using the fact that $H^2=-I$ we get

$$2H(\varphi H(\varphi))=-\varphi^2+H(\varphi)^2$$

[29]This, however, would require a proof. The function $\varphi^2-H(\varphi)^2$ does not belong to $\mathcal{S}(\mathbb{R})$ in general and we proved that the boundary value of the imaginary part is the Hilbert transform of the boundary value of the real part only in a specific situation when the real part is in $\mathcal{S}(\mathbb{R})$ and the extension is defined by the integral $F(z)$. We will not clarify this issue now as we will present in a moment a different, rigorous, proof based on the Fourier transform.

which implies the claim.

Now we present a different and rigorous proof. Let $m(\xi) = -i\,\mathrm{sgn}\,(\xi)$ be the symbol of the Hilbert transform. We have[30]

$$\begin{aligned}
&\widehat{\varphi^2}(\xi) + 2\big[H(\varphi H(\varphi))\big]^\wedge(\xi) = (\hat\varphi * \hat\varphi)(\xi) + 2m(\xi)(\hat\varphi * \widehat{H\varphi})(\xi)\\
&= \int_{-\infty}^{\infty} \hat\varphi(\zeta)\hat\varphi(\xi-\zeta)\,d\zeta + 2m(\xi)\int_{-\infty}^{\infty}\hat\varphi(\zeta)\hat\varphi(\xi-\zeta)m(\xi-\zeta)\,d\zeta = \heartsuit\,.
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \hat\varphi(\zeta)\hat\varphi(\xi-\zeta)m(\xi-\zeta)\,d\zeta = \int_{-\infty}^{\infty}\hat\varphi(\xi-\zeta)\hat\varphi(\zeta)m(\zeta)\,d\zeta$$

we have

$$\begin{aligned}
&2m(\xi)\int_{-\infty}^{\infty}\hat\varphi(\zeta)\hat\varphi(\xi-\zeta)m(\xi-\zeta)\,d\zeta\\
&= m(\xi)\int_{-\infty}^{\infty}\hat\varphi(\zeta)\hat\varphi(\xi-\zeta)\big(m(\zeta)+m(\xi-\zeta)\big)\,d\zeta
\end{aligned}$$

and hence

$$\heartsuit = \int_{-\infty}^{\infty}\hat\varphi(\zeta)\hat\varphi(\xi-\zeta)\big(1+m(\xi)(m(\zeta)+m(\xi-\zeta))\big) = \diamondsuit$$

Since $m(\xi) = -i\,\mathrm{sgn}\,(\xi)$ one easily verifies that

$$1 + m(\xi)(m(\zeta)+m(\xi-\zeta)) = m(\zeta)m(\xi-\zeta)$$

everywhere except $\xi = \zeta = 0$ and hence

$$\begin{aligned}
\diamondsuit &= \int_{-\infty}^{\infty}\hat\varphi(\zeta)\hat\varphi(\xi-\zeta)m(\zeta)m(\xi-\zeta)\,d\zeta\\
&= \int_{-\infty}^{\infty}\widehat{H\varphi}(\xi-\zeta)\widehat{H\varphi}(\zeta)\,d\zeta\\
&= \big(\widehat{H\varphi} * \widehat{H\varphi}\big)(\xi)\\
&= \big(H(\varphi)^2\big)^\wedge(\xi)\,.
\end{aligned}$$

Taking the inverse Fourier transform yields the result. □

Proof of Theorem 5.7. First observe that it suffices to prove the inequality

$$\|H\varphi\|_{p_k} \le C(p_k)\|\varphi\|_{p_k} \tag{5.12}$$

for $p = p_k = 2^k$, $k = 1,2,3,\dots$ Indeed, this and the duality argument (Theorem 4.6) will imply that

$$\|H\varphi\|_{p_k'} \le C(p_k)\|\varphi\|_{p_k'}$$

[30]In the arguments below we use Plancherel's theorem many times, because all functions for which we compute Hilbert's transform, Fourier's transform and convolution are in L^2 and not always in $\mathcal{S}(\mathbb{R})$.

and hence the Riesz-Thorin theorem will yield

$$\|H\varphi\|_p \le C(p)\|\varphi\|_p$$

for all $p_k' < p < p_k$. Since p_k can be arbitrarily large, boundedness of the Hilbert transform in L^p for all $1 < p < \infty$ will follow.

We already proved (5.12) for $k = 1$ with $C(p) = 1$ in Theorem 5.3. Suppose now that we established the inequality for $p = p_k$ and we will show how to deduce the inequality for $2p = p_{k+1}$. For $0 \neq \varphi \in \mathcal{S}(\mathbb{R})$ Cotlar's identity yields

$$\begin{aligned}
\|H\varphi\|_{2p} &= \|(H\varphi)^2\|_p^{1/2} \\
&\le \big(\|\varphi^2\|_p + \|2H(\varphi H(\varphi))\|_p\big)^{1/2} \\
&\le \big(\|\varphi\|_{2p}^2 + 2C(p)\|\varphi H(\varphi)\|_p\big)^{1/2} \\
&\le \big(\|\varphi\|_{2p}^2 + 2C(p)\|\varphi\|_{2p}\|H\varphi\|_{2p}\big)^{1/2}
\end{aligned}$$

and hence

$$\left(\frac{\|H\varphi\|_{2p}}{\|\varphi\|_{2p}}\right)^2 - 2C(p)\,\frac{\|H\varphi\|_{2p}}{\|\varphi\|_{2p}} - 1 \le 1\,.$$

This is a quadratic inequality which immediately yields

$$\frac{\|H\varphi\|_{2p}}{\|\varphi\|_{2p}} \le C(p) + \sqrt{C(p)^2 + 1}$$

and hence (5.12) for $p_{k+1} = 2p$ follows with

$$C(p_{k+1}) = C(p_k) + \sqrt{C(p_k)^2 + 1}\,.$$

The proof is complete. □

Remark. The proof gives also good estimates for the norm of the Hilbert transform in L^p.

5.3. L^p **multipliers.** Theorem 5.7 implies that $m(\xi) = -i\,\mathrm{sgn}\,(\xi)$ is an L^p multiplier $m \in \mathcal{M}_p(\mathbb{R})$, $1 < p < \infty$. Hence also

$$\chi_{[0,\infty)}(\xi) = (1 + i\,\mathrm{sgn}\,(\xi))/2 \in \mathcal{M}_p(\mathbb{R})\,.$$

Since translations, reflections and product of multipliers is a multiplier we conclude that

$$\chi_{[a,b]} \in \mathcal{M}_p(\mathbb{R})\,.$$

One dimensional multipliers generate n-dimensional ones. Indeed, if $m \in \mathcal{M}_p(\mathbb{R})$ and

$$\tilde{m}(\xi_1, \ldots, \xi_n) = m(\xi_i)\,, \tag{5.13}$$

the operator

$$T_{\tilde{m}}\varphi(x) = \big(T_m\varphi(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)\big)(x_i)$$

is bounded in L^p. This easily follows from the Fubini theorem

$$\begin{aligned}
&\int_{\mathbb{R}^n} |T_{\tilde{m}}\varphi(x)|^p\,dx \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |\big(T_m\varphi(x_1,\ldots,\cdot,\ldots,x_n)\big)(x_i)|^p\,dx_i \right) dx_1\ldots dx_{i-1}\,dx_{i+1}\ldots dx_n \\
&\le C\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |\varphi|^p\,dx_i \right) dx_1\ldots dx_{i-1}\,dx_{i+1}\ldots dx_n\,.
\end{aligned}$$

Thus $\tilde{m} \in \mathcal{M}_p(\mathbb{R}^n)$.

Corollary 5.12. *If* $m_1,\ldots,m_n \in \mathcal{M}_p(\mathbb{R})$*, then*

$$m(\xi_1,\ldots,\xi_n) = m_1(\xi_1)\ldots m_n(\xi_n) \in \mathcal{M}_p(\mathbb{R}^n)\,.$$

Proof. Indeed, m is a product of multipliers of the form (5.13). □

Since $\chi_{[0,\infty)} \in \mathcal{M}_p(\mathbb{R}^n)$, the characteristic function of the half-space $\mathbb{R}^n_+$ belongs to $\mathcal{M}_p(\mathbb{R}^n)$. Rotations, translations and product of such characteristic functions is also an L^p multiplier and hence we have.

Theorem 5.13. *The characteristic function of any convex polyhedron in* $\mathbb{R}^n$ *is an* L^p *multiplier for* $1 < p < \infty$*.*

5.4. **Pointwise convergence.** We will prove that $H^\varepsilon f \to Hf$ a.e. for $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. In this section we will treat the case $1 < p < \infty$ as the case $p = 1$ requires a different argument and will be discussed later.

Definition. The *maximal Hilbert transform* is the operator

$$H^*f(x) = \sup_{\varepsilon>0} |(H^\varepsilon f)(x)|$$

defined for all $f \in L^p(\mathbb{R})$, $1 \le p < \infty$.

Lemma 5.14. *For* $f \in L^p(\mathbb{R})$, $1 < p < \infty$ *and all* $x \in \mathbb{R}$ *we have*

$$(f * Q_\varepsilon)(x) = (H(f) * P_\varepsilon)(x)\,. \tag{5.14}$$

Proof. It suffices to prove the equality for $f = \varphi \in \mathcal{S}(\mathbb{R})$. Indeed, if $f \in L^p$ and $\varphi_k \in \mathcal{S}(\mathbb{R})$, $\varphi_k \to f$ in L^p, then $H(\varphi_k) \to H(f)$ in L^p and hence $\varphi_k * Q_\varepsilon(x) \to f * Q_\varepsilon(x)$, $H(\varphi_k) * P_\varepsilon(x) \to H(f) * P_\varepsilon(x)$ because of the fact that $P_\varepsilon, Q_\varepsilon \in L^{p'}$ and the Hölder inequality. Moreover since $Q_\varepsilon(x) = \varepsilon^{-1}Q_1(x/\varepsilon)$ and $P_\varepsilon(x) = \varepsilon^{-1}P_1(x/\varepsilon)$ it suffices to prove the equality for $\varepsilon = 1$.

Taking the Fourier transform of (5.14) for with $f = \varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon = 1$ we see that it is equivalent to[31]

$$\hat{\varphi}(\xi)\hat{Q}_1(\xi) = -i\,\mathrm{sgn}\,(\xi)\hat{\varphi}(\xi)e^{-2\pi|\xi|}$$

[31] See Corollary 2.24.

Observe that this identity follows from

$$\left(-i\,\mathrm{sgn}\,(\cdot)e^{-2\pi|\cdot|}\right)^{\vee}(x)=\frac{1}{\pi}\frac{x}{x^2+1}\,. \tag{5.15}$$

The proof of (5.15) goes as follows

$$\begin{aligned}\left(-i\,\mathrm{sgn}\,(\cdot)e^{-2\pi|\cdot|}\right)^{\vee}(x) &= -i\int_{-\infty}^{\infty}\mathrm{sgn}\,(\xi)e^{-2\pi|\xi|}e^{2\pi i x\xi}\,d\xi\\ &= -i\int_0^{\infty}\left(e^{-2\pi\xi}e^{2\pi i x\xi}-e^{-2\pi\xi}e^{2\pi x(-\xi)}\right)d\xi\\ &= 2\int_0^{\infty}e^{-2\pi\xi}\sin(2\pi x\xi)\,d\xi\\ &= \frac{1}{\pi}\frac{x}{x^2+1}\,,\end{aligned}$$

where the last equality follows from the twice integration by parts. □

Lemma 5.15 (Cotlar). *If $f\in L^p(\mathbb{R})$, $1<p<\infty$, then for all $x\in\mathbb{R}$ we have*

$$|H^*f(x)|\leq\mathcal{M}f(x)+\mathcal{M}(Hf)(x)\,.$$

Proof. Following notation from the proof of Theorem 5.4 we have

$$f*Q_\varepsilon(x)-H^\varepsilon f(x)=\frac{1}{\pi}(f*\psi_\varepsilon)(x)$$

and the function

$$\Psi(x)=\frac{1}{x^2+1}$$

is an integrable radially decreasing majorant of ψ. Hence Theorem 3.12 yields

$$|f*Q_\varepsilon(x)-H^\varepsilon f(x)|\leq\frac{1}{\pi}\|\Psi\|_1\mathcal{M}f(x)=\mathcal{M}f(x)$$

Another application of the same theorem combined with Lemma 5.14 gives

$$\sup_{\varepsilon>0}|f*Q_\varepsilon(x)|=\sup_{\varepsilon>0}|H(f)*P_\varepsilon(x)|\leq\|P_1\|_1\mathcal{M}(Hf)(x)=\mathcal{M}(Hf)(x)\,.$$

Thus

$$|H^*f(x)|\leq\sup_{\varepsilon>0}|f*Q_\varepsilon(x)|+\mathcal{M}f(x)\leq\mathcal{M}(Hf)(x)+\mathcal{M}f(x)\,.$$

The proof is complete. □

Corollary 5.16. *The operator H^* is of strong type (p,p) for all $1<p<\infty$, i.e. for $f\in L^p(\mathbb{R})$*

$$\|H^*f\|_p\leq C_p\|f\|_p\,.$$

Proof. It follows immediately from Lemma 5.15, boundedness of the Hilbert transform in L^p (Theorem 5.7) and boundedness of the maximal function in L^p (Theorem 3.10). □

Corollary 5.17. *For $f \in L^p(\mathbb{R})$, $1 < p < \infty$, $H^\varepsilon f \to Hf$ as $\varepsilon \to 0$ both in L^p and a.e.*

Remark. Compare the proof with that of Theorem 1.2.

Proof. If $\varphi \in \mathcal{S}(\mathbb{R})$, then $H^\varepsilon \varphi \to H\varphi$ everywhere. Since

$$|H^\varepsilon \varphi - H\varphi| \leq H^*\varphi + |H\varphi| \in L^p$$

we conclude that

$$H^\varepsilon \varphi \to H\varphi \quad \text{in } L^p. \tag{5.16}$$

Now we will prove that for $f \in L^p$, $H^\varepsilon f$ converges a.e. to some measurable function g. To this end it suffices to show that

$$\Omega f(x) = 0 \quad \text{a.e.} \tag{5.17}$$

where

$$\Omega f(x) = \limsup_{\varepsilon \to 0} H^\varepsilon f(x) - \liminf_{\varepsilon \to 0} H^\varepsilon f(x).$$

Note that

$$0 \leq \Omega f(x) \leq 2H^* f(x)$$

and hence

$$|\{x : \Omega f(x) > t\}| \leq \frac{2}{t^p} \int_{\mathbb{R}} |H^* f|^p \leq \frac{c}{t^p} \int_{\mathbb{R}} |f|^p \, .$$

To prove (5.17) it suffices to show that for any $t > 0$

$$|\{x : \, \Omega f(x) > t\}| = 0 \, . \tag{5.18}$$

Given any $\gamma > 0$ let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\|f - \varphi\|_p < \gamma t$. It is easy to see that

$$\Omega f \leq \Omega(f - \varphi) + \Omega \varphi = \Omega(f - \varphi) \, ,$$

where the last equality follows from the fact that $H^\varepsilon \varphi \to H\varphi$ everywhere. Thus

$$|\{x : \, \Omega f(x) > t\}| \leq |\{x : \, \Omega(f - \varphi)(x) > t\}| \leq \frac{c}{t^p} \int_{\mathbb{R}} |f - \varphi|^p \leq c\gamma^p \, .$$

It is true for any $\gamma > 0$, so (5.18) and hence (5.17) follows. Since $|H^\varepsilon f| \leq H^* f$ and $H^\varepsilon f \to g$ a.e. we conclude that $|g| \leq H^* f$ a.e. Now

$$|H^\varepsilon f - g| \leq 2H^* f \in L^p$$

and the dominated convergence theorem yields $H^\varepsilon f \to g$ in L^p. It remains to prove that $g = Hf$ a.e.

Given $\gamma > 0$ let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\|f - \varphi\|_p < \gamma$ and let $\varepsilon > 0$ be such that

$$\|H\varphi - H^\varepsilon \varphi\|_p < \gamma \quad \text{(see (5.16))}$$

$$\|H^\varepsilon f - g\|_p < \gamma \, .$$

We have

$$\begin{aligned}\|Hf-g\|_p &\leq \|H(f-\varphi)\|_p+\|H\varphi-H^{\varepsilon}\varphi\|_p+\|H^{\varepsilon}(f-\varphi)\|_p+\|H^{\varepsilon}f-g\|_p\\ &\leq C\|f-\varphi\|_p+\gamma+\|H^*(f-\varphi)\|_p+\gamma\leq C'\gamma\end{aligned}$$

and hence $g=Hf$ a.e. □

The proofs of Theorem 1.2 and Corollary 5.17 are based on the same method. Not surprisingly, the same argument appears in other similar situations, so it is wise to present it in a form of an abstract and general result which will allow us to apply the result directly and avoid repeating the same argument over and over again.

Theorem 5.18. *Let (X,μ) be a measure space and let $\{T_t\}_{t>0}$ be a family of linear operators from $L^p(\mu)$, $1\leq p<\infty$ into the space of measurable functions on X. Suppose that the limit*

$$\lim_{t\to 0} T_t f(x) \tag{5.19}$$

exists a.e. for all functions f in a dense subset $\mathcal{A}\subset L^p(\mu)$. Define the maximal operator associated with the family $\{T_t\}$ by

$$T^*f(x)=\sup_{t>0}|T_tf(x)|\,.$$

If T^* is of weak type (p,q), $1\leq q<\infty$, then the limit (5.19) exists a.e. for all $f\in L^p(\mu)$. Denote the limit by

$$Tf(x)=\lim_{t\to 0}T_tf(x)\quad a.e.$$

for all $f\in L^p(\mu)$. If in addition T^* is of strong type (p,q), then T is of strong type (p,q) and

$$T_tf\to Tf\qquad in\ L^q(\mu)\ as\ t\to 0$$

for all $f\in L^p(\mu)$.

Proof. We can assume that the functions are real valued as otherwise we can consider the real and imaginary parts separately. Suppose T is of weak type (p,q). We will prove now the first part of the theorem which says that T_tf converges a.e. as $t\to 0$ for all $f\in L^p(\mu)$. To this end it suffices to show that $\Omega f=0$ a.e., where

$$\Omega f(x)=\limsup_{t\to 0}T_tf(x)-\liminf_{t\to 0}T_tf(x)\,.$$

Note that $0\leq\Omega f(x)\leq 2T^*f(x)$ and hence

$$|\{x\in X:\,\Omega f(x)>t\}|\leq\left(\frac{C\|f\|_p}{t}\right)^q\,.$$

For $\varepsilon>0$ let $\varphi\in\mathcal{A}$ be such that $\|f-\varphi\|_p<\varepsilon t$. It is easy to see that

$$\Omega f\leq\Omega(f-\varphi)+\Omega\varphi=\Omega(f-\varphi)\,,$$

because $\Omega\varphi = 0$ a.e. since $T_t\varphi$ converges a.e. as $t \to 0$. Thus

$$|\{\Omega f > t\}| \le |\{\Omega(f-\varphi) > t\}| \le \left(\frac{C\varepsilon t}{t}\right)^q = C^q\varepsilon^q\,.$$

Since it is true for any $t > 0$ and $\varepsilon > 0$ we conclude that $\Omega f = 0$ a.e. which completes the proof of the first part of the theorem.

Suppose now that T^* is of strong type (p, q). Since $|T_t f| \le T^* f$ we conclude that $|Tf| \le T^* f$ and hence

$$|T_t f - Tf| \le 2T^* f \in L^q\,.$$

This inequality, the fact that $T_t f \to Tf$ a.e. and the dominated convergence theorem imply that $T_t f \to Tf$ in L^q. □

6. The Riesz transforms

The Hilbert transform is, up to a constant, convolution with the principal value of $1/x = x/|x|^2$. Thus a natural generalization to the n-dimensional case would be convolution with the principal value of $x/|x|^{n+1}$. However, $x/|x|^{n+1}$ is a vector valued function, so we should consider its components $x_j/|x|^{n+1}$ separately. For $\varphi \in \mathcal{S}_n$ we define

$$\begin{aligned} W_j[\varphi] &= c_n\,\text{p.v.}\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+1}}\,\varphi(x)\,dx \\ &= c_n \lim_{\varepsilon\to 0}\int_{|x|\ge\varepsilon} \frac{x_j}{|x|^{n+1}}\,\varphi(x)\,dx\,, \end{aligned}$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$. Note that the constant is the same as the one in the definition of the Poisson kernel.

Exercise. *Prove that the above limit exists for any $\varphi \in \mathcal{S}_n$ and that $W_j \in \mathcal{S}_n'$.*

Definition. For $1 \le j \le n$ the *Riesz transform* of a function f is defined by

$$\begin{aligned} (R_j f)(x) &= (W_j * f)(x) \\ &= c_n\,\text{p.v.}\int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}}\,f(x-y)\,dy \\ &= c_n \lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon} \frac{x_j-y_j}{|x-y|^{n+1}}\,f(y)\,dy\,. \end{aligned}$$

The Riesz transform is well defined for $\varphi \in \mathcal{S}_n$ and the question is how to extend it to $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

The following results generalizes Theorem 5.2.

Theorem 6.1.

$$\widehat{W_j}(\xi) = -i\,\frac{\xi_j}{|\xi|}\,.$$

Proof. For $\varphi \in \mathcal{S}_n$ we have

$$\begin{aligned}
\widehat{W}_j[\varphi] &= W_j[\hat{\varphi}] = \lim_{\varepsilon\to 0} c_n \int_{|\xi|\geq\varepsilon} \hat{\varphi}(\xi)\, \frac{\xi_j}{|\xi|^{n+1}}\, d\xi \\
&= \lim_{\varepsilon\to 0} c_n \int_{\varepsilon\leq|\xi|\leq\varepsilon^{-1}} \left(\int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi}\, dx\right) \frac{\xi_j}{|\xi|^{n+1}}\, d\xi \\
&= \lim_{\varepsilon\to 0} c_n \int_{\mathbb{R}^n} \varphi(x) \Big(\underbrace{\int_{\varepsilon\leq|\xi|\leq\varepsilon^{-1}} e^{-2\pi i x\cdot\xi}\, \frac{\xi_j}{|\xi|^{n+1}}\, d\xi}_{I} \Big)\, dx = \heartsuit\,.
\end{aligned}$$

Expressing the integral I is spherical coordinates we have

$$\begin{aligned}
I &= \int_\varepsilon^{\varepsilon^{-1}} s^{n-1} \left(\int_{S^{n-1}} e^{-2\pi i x\cdot(s\theta)}\, \frac{s\theta_j}{s^{n+1}}\, d\theta\right) ds \\
&= \int_\varepsilon^{\varepsilon^{-1}} \int_{S^{n-1}} \big(\cos(2\pi s x\cdot\theta) - i\sin(2\pi s x\cdot\theta)\big)\theta_j\, d\sigma(\theta)\, \frac{ds}{s} \\
&= -i \int_\varepsilon^{\varepsilon^{-1}} \int_{S^{n-1}} \sin(2\pi s x\cdot\theta)\theta_j\, d\sigma(\theta)\, \frac{ds}{s} \\
&= -i \int_{S^{n-1}} \left(\int_\varepsilon^{\varepsilon^{-1}} \frac{\sin(2\pi s|x\cdot\theta|)}{s}\, ds\right) \operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta) \\
&= -i \int_{S^{n-1}} \left(\int_{2\pi|x\cdot\theta|\varepsilon}^{2\pi|x\cdot\theta|\varepsilon^{-1}} \frac{\sin t}{t}\, dt\right) \operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta)\,.
\end{aligned}$$

Thus

$$\heartsuit = \int_{\mathbb{R}^n} \varphi(x) \left(-ic_n\frac{\pi}{2} \int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta)\right) dx\,.$$

Indeed, we could pass to the limit under the sign of the integral using the dominated convergence theorem and we employed the equality

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}\,.$$

Thus it remains to prove that

$$c_n\frac{\pi}{2} \int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta) = \frac{x_j}{|x|}\,. \tag{6.1}$$

We will need the following result.

Lemma 6.2. *If $m : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function that is homogeneous of degree 0, i.e. $m(tx) = m(x)$ for $t > 0$, and commutes with the orthogonal transformations, i.e.*

$$m(\rho(x)) = \rho(m(x)) \tag{6.2}$$

for all $x \in \mathbb{R}^n$ and $\rho \in O(n)$, then there is a constant c such that

$$m(x) = c\,\frac{x}{|x|} \quad \textit{for all } x \neq 0. \tag{6.3}$$

Before we will prove the lemma we show how to use it to establish (6.1). it is obvious that the function $m : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$m(x) = \int_{S^{n-1}} \operatorname{sgn}(x \cdot \theta)\theta \, d\sigma(\theta)$$

is homogeneous of degree 0. It also commutes with orthogonal transformations since

$$\begin{aligned} m(\rho(x)) &= \int_{S^{n-1}} \operatorname{sgn}(\rho(x) \cdot \theta)\theta \, d\sigma(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(x \cdot \rho^{-1}(\theta))\theta \, d\sigma(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(x \cdot \theta)\rho(\theta) \, d\sigma(\theta) && (6.4) \\ &= \rho\left(\int_{S^{n-1}} \operatorname{sgn}(x \cdot \theta)\theta \, d\sigma(\theta)\right) && (6.5) \\ &= \rho(m(x)) \,. && (6.6) \end{aligned}$$

Note that (6.5) follows from the fact that ρ induces a volume preserving change of variables on S^{n-1}, while (6.6) is a direct consequence of linearity of ρ. thus the lemma yields

$$\int_{S^{n-1}} \operatorname{sgn}(x \cdot \theta)\theta \, d\sigma(\theta) = c \, \frac{x}{|x|}$$

and hence looking at the jth component we have

$$\int_{S^{n-1}} \operatorname{sgn}(x \cdot \theta)\theta_j \, d\sigma(\theta) = c \, \frac{x_j}{|x|} \,. \tag{6.7}$$

now it remains to prove that

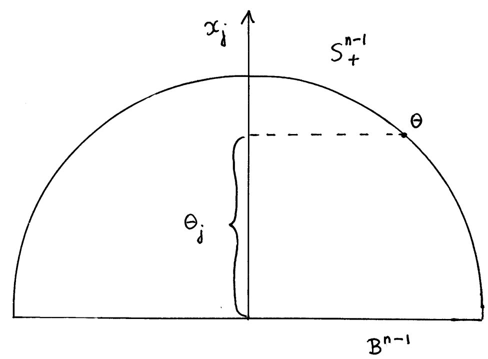
$$c = \left(c_n \, \frac{\pi}{2}\right)^{-1} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} = 2\omega_{n-1} \,.$$

Taking $x = e_j$ in (6.7) we have

$$\int_{S^{n-1}} |\theta_j| \, d\sigma(\theta) = c \,.$$

The unit ball B^{n-1} in coordinates perpendicular to x_j split the sphere S^{n-1} into two half spheres $S^{n-1}_{\pm}$. Thus

$$c = 2 \int_{S^{n-1}_{+}} \theta_j \, d\sigma(\theta) \,.$$



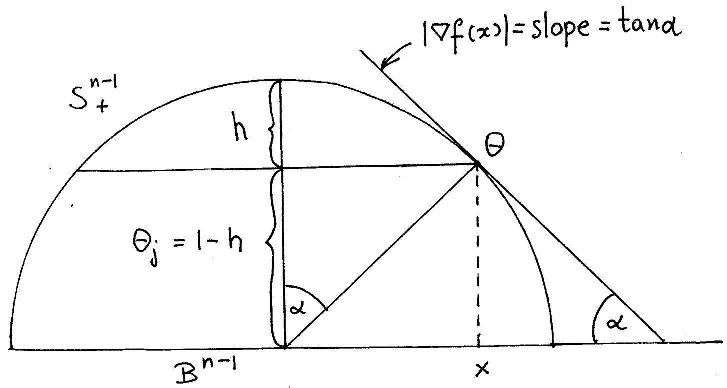
Recall that if $M\subset\mathbb{R}^n$ is a graph of a C^1 function $f:\Omega\to\mathbb{R}$, $\Omega\subset\mathbb{R}^{n-1}$, then for a measurable function g on M we have

$$\int_M g\,d\sigma=\int_\Omega g(x,f(x))\sqrt{1+|\nabla f(x)|^2}\,dx\,.$$

In our situation we parametrize S^{n-1}_+ as a graph of the function

$$f(x)=\sqrt{1-|x|^2}\,,\quad x\in B^{n-1}\,.$$

Form the picture



we conclude

$$\begin{aligned}\int_{S^{n-1}_+}\theta_j\,d\sigma(\theta) &= \int_{S^{n-1}_+}(1-h)\,d\sigma(\theta)=\int_{B^{n-1}}(1-h)\sqrt{1+\tan^2\alpha}\,dx\\ &= \int_{B^{n-1}}dx=\omega_{n-1}\,,\end{aligned}$$

because

$$\sqrt{1+\tan^2\alpha}=\frac{1}{\cos\alpha}=\frac{1}{1-h}$$

and the result follows. Thus we are left with the proof of the lemma.

Proof of Lemma 6.2. Let $e_1, e_2, \ldots, e_n$ be the standard orthogonal basis of $\mathbb{R}^n$. If $[\rho_{jk}]$ is the matrix representation of $\rho \in O(n)$, then the condition (6.2) reads as

$$m_j(\rho(x)) = \sum_{k=1}^{n} \rho_{jk} m_k(x), \quad j = 1, 2, \ldots, n, \tag{6.8}$$

where $m(x) = (m_1(x), \ldots, m_n(x))$.

Let $m_1(e_1) = c$. Consider all $\rho \in O(n)$ such that $\rho(e_1) = e_1$. This condition means that the first column of the matrix $[\rho_{jk}]$ equals e_1, i.e. $\rho_{11} = 1$, $\rho_{j1} = 0$, for $j > 1$. Since columns are orthogonal, for $k > 1$ we have

$$0 = \sum_{j=1}^{n} \rho_{j1} \rho_{jk} = \rho_{1k}\,.$$

Thus

$$\rho = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \rho_{22} & \ldots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \rho_{n2} & \ldots & \rho_{nn} \end{bmatrix},$$

where $[\rho_{jk}]_{j,k=2}^n$ is the matric of an arbitrary orthogonal transformation in the $(n-1)$-dimensional subspace orthogonal to e_1.

For $x = e_1 = \rho(e_1) = \rho(x)$ and $j \geq 2$ identity (6.8) yields

$$m_j(e_1) = \sum_{k=1}^{n} \rho_{jk} m_k(e_1) = \sum_{k=2}^{n} \rho_{jk} m_k(e_1)\,,$$

and hence

$$\begin{bmatrix} m_2(e_1) \\ \vdots \\ m_n(e_1) \end{bmatrix} = \begin{bmatrix} \rho_{22} & \ldots & \rho_{2n} \\ \vdots & \ddots & \vdots \\ \rho_{n2} & \ldots & \rho_{nn} \end{bmatrix} \begin{bmatrix} m_2(e_1) \\ \vdots \\ m_n(e_1) \end{bmatrix}.$$

That means the vector $[m_2(e_1), \ldots, m_n(e_1)]^T$ is fixed under an arbitrary orthogonal transformation of $\mathbb{R}^{n-1}$, so it must be a zero vector, i.e.

$$m_2(e_1) = \ldots = m_n(e_1) = 0\,.$$

Now formula (6.8) for any $\rho \in O(n)$ and $x = e_1$, takes the form

$$m_j(\rho(e_1)) = \rho_{j1} m_1(e_1) = c\rho_{j1}\,.$$

By homogeneity it suffices to prove (6.3) for $|x| = 1$. Let $\rho \in O(n)$ be such that $\rho(e_1) = x$. Then $\rho_{j1} = x_j$, $j = 1, 2, \ldots, n$ and hence

$$m_j(x) = c\rho_{j1} = cx_j = c\,\frac{x_j}{|x|}\,.$$

This completes the proof of the lemma and hance that of Theorem 6.1. $\square$

Corollary 6.3. *For $\varphi \in \mathcal{S}_n$ and $1 \leq j \leq n$*

$$(R_j\varphi)(x) = \left(-\frac{i\xi_j}{|\xi|}\,\hat{\varphi}(\xi)\right)^{\vee}(x)\,.$$

Since the function $m(\xi) = -i\xi_j/|\xi|$ is bounded, $R_j \in \mathcal{M}^{2,2}$ and

$$\|R_j f\|_2 \leq \|f\|_2 \quad \text{for } f \in L^2(\mathbb{R}^n).$$

Moreover since the Riesz transform is a convolution with a tempered distribution, for every $\varphi \in \mathcal{S}_n$, $R_j\varphi \in C^\infty$ is slowly increasing and all its derivatives are slowly increasing.

Corollary 6.4. *The Riesz transforms satisfy*

$$\sum_{j=1}^n R_j^2 = -I \quad \textit{on } L^2(\mathbb{R}^n).$$

Proof. Applying the previous corollary and the Plancherel theorem, for $f \in L^2$ we have

$$\left(\sum_{j=1}^n R_j^2 f\right)^{\wedge}(\xi) = \sum_{j=1}^n \left(-\frac{i\xi_j}{|\xi|}\right)^2 \hat{f}(\xi) = -\hat{f}(\xi)$$

which yields the claim. □

An amazing property of the Riesz transforms is that they allow to compute mixed partial derivatives $\partial_j\partial_k u$ is we only know Δu. More precisely we have

Proposition 6.5. *If $\varphi \in \mathcal{S}_n$, then for $1 \leq j,k \leq n$ we have*

$$\frac{\partial\varphi}{\partial x_j \partial x_k} = -R_jR_k\Delta\varphi(x)\,.$$

Proof. For $\varphi \in \mathcal{S}_n$ we have

$$\begin{aligned}
(\partial_j\partial_k\varphi)^{\wedge}(\xi) &= (2\pi i\xi_j)(2\pi i\xi_k)\hat{\varphi}(\xi) \\
&= -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)(-4\pi^2|\xi|^2)\hat{\varphi}(\xi) \\
&= -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\widehat{\Delta\varphi}(\xi) \\
&= (-R_jR_k\Delta\varphi)^{\wedge}(\xi)
\end{aligned}$$

and the result follows by taking the inverse Fourier transform in L^2. □

It is quite convincing the the above argument applied to $u \in \mathcal{S}_n'$ such that $\Delta u = f \in L^2$, gives $\partial_j\partial_k u = -R_jR_k f$. However this is not true. For example if $u = xy$, then as a slowly increasing function $u \in \mathcal{S}_2'$. Clearly $\Delta u = 0 = f$, but $\partial_x\partial_y u = 1 \neq 0 = -R_xR_y f$. In general we have

Theorem 6.6. *If $u \in \mathcal{S}'_n$ satisfies*

$$\Delta u = f \in L^2(\mathbb{R}^n)\,, \tag{6.9}$$

then for any $1 \le j, k \le n$

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_j R_k f + P_{jk}\,,$$

where P_{jk} is a polynomial.

Proof. Taking the Fourier transform of (6.9) we have

$$-4\pi^2|\xi|^2\hat{u} = \hat{f}\,.$$

Hence if $\lambda \in C^\infty(\mathbb{R}^n)$ and all its derivatives are slowly increasing, then

$$-\lambda(\xi)(-4\pi^2|\xi|^2)\hat{u} = -\lambda(\xi)\hat{f}\,. \tag{6.10}$$

According to Corollary 2.51 it suffices to prove that the tempered distribution

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k} + R_j R_k f\right)^\wedge$$

has support contained in $\{0\}$. Let $\varphi \in \mathcal{S}_n$ be such that $0 \notin \operatorname{supp}\varphi$, say $\varphi(x) = 0$ for $|x| \le r$. We need to show that

$$(\partial_j \partial_k u)^\wedge[\varphi] = (-R_j R_k f)^\wedge[\varphi]\,.$$

Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 0$ for $|x| \le r/2$ and $\eta(x) = 1$ for $|x| \ge r$. The function η and its derivatives are slowly increasing and $\eta\varphi = \varphi$. Hence

$$\begin{aligned}(\partial_j \partial_k u)^\wedge[\varphi] &= \big((2\pi i \xi_i)(2\pi i \xi_j)\hat{u}\big)[\eta\varphi] \\ &= \big(\eta(\xi)(2\pi i \xi_i)(2\pi i \xi_j)\hat{u}\big)[\eta\varphi] = \heartsuit\end{aligned}$$

Observe that

$$\eta(\xi)(2\pi i\xi_i)(2\pi i \xi_j) = -\underbrace{\eta(\xi)\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)}_{\lambda(\xi)}(-4\pi^2|\xi|^2)$$

and $\lambda \in C^\infty$ and its derivatives are slowly increasing, since η vanishes in a neighborhood of $\xi = 0$. Thus

$$\begin{aligned}\heartsuit &= \big(-\lambda(\xi)(-4\pi^2|\xi|^2)\hat{u}\big)[\varphi] \\ &= (-\lambda(\xi)\hat{f}(\xi))[\varphi] \\ &= \left(-\eta(\xi)\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f}(\xi)\right)[\varphi] \\ &= \left(-\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f}(\xi)\right)[\varphi] \\ &= (-R_j R_k f)^\wedge[\varphi]\,.\end{aligned}$$

The proof is complete. □

Roughly speaking, one dimensional directional sections of the kernel of the Riesz transform defines one dimensional Hilbert transforms and it is possible to use this fact to prove boundedness of R_j in L^p by a so called *method of rotations*. Since the same method works for a larger class of operators we postpone the proof of the boundedness of R_j in L^p to the next section where a more general result will be proved.

6.1. **Homogeneous distributions.** In this section we will present a different and shorter proof of Theorem 6.1.

Recall that a function f is homogeneous of degree a if for all $0 \neq x \in \mathbb{R}^n$ and $t > 0$

$$f(tx) = t^a f(x).$$

For such a function and $\varphi \in \mathcal{S}_n$ we have

$$\int_{\mathbb{R}^n} f(x)\varphi_t(x)\, dx = t^a \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx\,,$$

where $\varphi_t(x) = t^{-n}\varphi(x)$. This suggests the following definition.

DEFINITION. We say that a distribution $u \in \mathcal{S}_n'$ is *homogeneous of degree a* if for any $\varphi \in \mathcal{S}_n$

$$u[\varphi_t] = t^a u[\varphi]\,.$$

Proposition 6.7. *If $u \in \mathcal{S}'$ is homogeneous of degree a, then $\hat{u}$ is homogeneous of degree $-n-a$.*

Proof. Recall that for $\varphi \in \mathcal{S}_n$ we have

$$\widehat{\varphi_t}(\xi) = \hat{\varphi}(t\xi) \quad \text{and} \quad (\hat{\varphi})_{t^{-1}} = t^n \hat{\varphi}(t\xi)\,,$$

so

$$\widehat{\varphi_t}(\xi) = t^{-n}(\hat{\varphi})_{t^{-1}}(\xi)\,.$$

Thus

$$\hat{u}[\varphi_t] = u[\widehat{\varphi_t}] = t^{-n}u[(\hat{\varphi})_{t^{-1}}] = t^{-n-a}u[\hat{\varphi}] = t^{-n-a}\hat{u}[\varphi]\,.$$

The proof is complete. □

As an application of this result we will prove the following

Proposition 6.8. *For $n/2 < a < n$*

$$\left(|x|^{-a}\right)^\wedge(\xi) = \frac{\pi^{a-\frac{n}{2}}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}|\xi|^{a-n}\,.$$

Proof. Observe that $|x|^{-a} \in L^1 + L^2$. Indeed,

$$|x|^{-a}\chi_{\{|x|\leq 1\}} \in L^1 \quad \text{and} \quad |x|^{-a}\chi_{\{|x|>1\}} \in L^2\,,$$

so the Fourier transform of $|x|^{-a}$ is function in $C_0 + L^2$. Since $|x|^{-a}$ is radially symmetric, the fact that the Fourier transform commutes with rotations implies that its Fourier transform is radially symmetric too. Lastly, $|x|^{-a}$ is homogeneous of degree $-a$, so Proposition 6.7 implies that the Fourier transform is homogeneous of degree $-n+a$. Thus

$$\left(|x|^{-a}\right)^{\wedge}(\xi) = c_{a,n}|\xi|^{a-n}$$

and it remains to compute the coefficient $c_{a,n}$. Employing the fact that $e^{-\pi|x|^2}$ is a fixed point of the Fourier transform we have

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{-a}\,dx = c_{a,n}\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{a-n}\,dx\,. \tag{6.11}$$

The integrals in this identity are easy to compute. Indeed, for $\gamma > -n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{\gamma}\,dx &= \int_0^\infty s^{n-1}|S^{n-1}|e^{-\pi s^2}s^\gamma\,ds \\
&= n\omega_n\int_0^\infty e^{-\pi s^2}s^{n+\gamma-1}\,ds \\
&\overset{t=\pi s^2}{=} \frac{n\omega_n}{2\pi^{\frac{n+\gamma}{2}}}\int_0^\infty e^{-t}t^{\frac{n+\gamma}{2}-1}\,dt \\
&= \frac{n\omega_n}{2\pi^{\frac{n+\gamma}{2}}}\Gamma\left(\frac{n+\gamma}{2}\right) = \frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{n}{2}\right)}\,.
\end{aligned}$$

Applying this formula to both sides of (6.11) we have

$$\frac{\Gamma\left(\frac{n-a}{2}\right)}{\pi^{-\frac{a}{2}}\Gamma\left(\frac{n}{2}\right)} = c_{a,n}\,\frac{\Gamma\left(\frac{a}{2}\right)}{\pi^{\frac{a-n}{2}}\Gamma\left(\frac{n}{2}\right)}$$

and hence

$$c_{a,n} = \pi^{a-\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\,.$$

The proof is complete. □

The distribution W_j arises naturally as an attempt to differentiate the function $1/|x|^{n-1}$. Namely we have

Proposition 6.9. *If $n \geq 2$, then $|x|^{1-n} \in \mathcal{S}_n'$ and its distributional partial derivatives satisfy*

$$\left(\frac{\partial}{\partial x_j}|x|^{1-n}\right)[\varphi] = (1-n)\,\text{p.v.}\,\frac{x_j}{|x|^{n+1}}\,.$$

Before we prove the proposition let us recall the integration by parts formula for functions defined in a domain in $\mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary and $f, g \in C^1(\overline{\Omega})$, then

$$\int_\Omega (\nabla f(x)g(x) + f(x)\nabla g(x))\,dx = \int_{\partial\Omega} fg\vec{\nu}\,d\sigma\,, \tag{6.12}$$

where $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ is the unit outer normal vector to $\partial\Omega$. Comparing jth components on both sides of (6.12) we have

$$\int_\Omega \frac{\partial f}{\partial x_j}(x)\, g(x)\, dx = -\int_\Omega f(x)\, \frac{\partial g}{\partial x_j}(x)\, dx + \int_{\partial\Omega} f g\, \nu_j\, d\sigma\,. \tag{6.13}$$

Proof of Proposition 6.9. Let $A(\varepsilon, R) = \{x : \varepsilon \le |x| \le R\}$. We have

$$\begin{aligned}
\left(\frac{\partial}{\partial x_j}|x|^{1-n}\right)[\varphi] &= -\int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_j}\, |x|^{1-n}\, dx \\
&= \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty} -\int_{\varepsilon \le |x| \le R} \frac{\partial \varphi}{\partial x_j}\, |x|^{1-n}\, dx\right) \\
&= \lim_{\varepsilon\to 0}\Big(\lim_{R\to\infty}\Big(\int_{\varepsilon\le|x|\le R} \varphi(x)\, \underbrace{\frac{\partial}{\partial x_j}\, |x|^{1-n}}_{(1-n)x_j/|x|^{n+1}}\, dx + \int_{\partial A(\varepsilon,R)} \varphi(x)|x|^{1-n}\, \nu_j\, d\sigma\Big)\Big) \\
&= \lim_{\varepsilon\to 0}\Big((1-n)\int_{|x|\ge\varepsilon} \varphi(x)\, \frac{x_j}{|x|^{n+1}} + \int_{|x|=\varepsilon} \varphi(x)|x|^{1-n}\, \nu_j\, d\sigma\Big)
\end{aligned}$$

Indeed, we could pass to the limit with $R \to \infty$, because the part of the second integral corresponding to the integration over $\{|x| = R\}$ clearly converges to zero. Since the integral of the function $|x|^{1-n}\,\nu_j$ over the sphere $|x| = \varepsilon$ equals zero we have

$$\int_{|x|=\varepsilon} \varphi(x)|x|^{1-n}\,\nu_j\, d\sigma = \int_{|x|=\varepsilon} (\varphi(x) - \varphi(0))|x|^{1-n}\,\nu_j\, d\sigma \to 0 \quad \text{as } \varepsilon \to 0,$$

because

$$\big|\big(\varphi(x) - \varphi(0)\big)|x|^{1-n}\,\nu_j\big| \le C\varepsilon^n\,,$$

and the result follows. □

Proof of Theorem 6.1. Applying Proposition 6.9 and then Proposition 6.8 we have

$$\begin{aligned}
\left(\text{p.v.}\,\frac{x_j}{|x|^{n+1}}\right)^\wedge &= \frac{1}{1-n}\left(\frac{\partial}{\partial x_j}\,|x|^{1-n}\right)^\wedge \\
&= \frac{2\pi i \xi_j}{1-n}\,\big(|x|^{1-n}\big)^\wedge \\
&= \frac{2\pi i \xi_j}{1-n}\,\pi^{n-1-\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-(n-1)}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\,|\xi|^{-1} \\
&= \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,\frac{-i\xi_j}{|\xi|}\,,
\end{aligned}$$

Where we used facts that

$$\Gamma(1/2) = \pi^{1/2} \quad \text{and} \quad \frac{n-1}{2}\Gamma\left(\frac{n-1}{2}\right) = \Gamma\left(\frac{n+1}{2}\right)\,.$$

The proof is complete. □

6.2. **The Bochner-Hecke formula.** Now we will generalize Theorem 6.1. The distribution W_j is, up to a constant, the principal value of $x_j/|x|^{n+1}$. The function x_j is harmonic and thus Theorem 6.1 follows also from a more general formula for the Fourier transform of

$$\text{p.v.}\,\frac{P_k(x)}{|x|^{n+k}}\,,$$

where $P_k(x)$ is a homogeneous harmonic polynomial of degree $k \geq 1$.

Definition. We say that $P_k(x)$ is a *homogeneous harmonic polynomial of degree k* if

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha \quad \text{and} \quad \Delta P_k = 0\,.$$

Let us start with a general observation. For $\Omega \in L^1(S^{n-1})$ with

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0 \tag{6.14}$$

we define

$$K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^n}\,, \quad x \neq 0\,.$$

As in the case of the Riesz transform $K_\Omega \notin L^1$, so in order to define K_Ω as a tempered distribution we need to consider the principal value of the integral. For $\varphi \in \mathcal{S}_n$ we define

$$\begin{aligned} W_\Omega[\varphi] &= \text{p.v.} \int_{\mathbb{R}^n} K_\Omega(x)\varphi(x)\, dx \\ &= \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} K_\Omega(x)\varphi(x)\, dx \\ &= \lim_{\varepsilon\to 0} \int_{\varepsilon\leq|x|\leq\varepsilon^{-1}} K_\Omega(x)\varphi(x)\, dx \end{aligned}$$

As in the case of Theorem 5.1 one can prove that for $\varphi \in \mathcal{S}_n$ the limit exists and defines $W_\Omega \in \mathcal{S}_n'$. Note that the condition (6.14) plays an essential role in the proof.

If P_k is a homogeneous harmonic polynomial of degree $k \geq 1$ we can write

$$\frac{P_k(x)}{|x|^{n+k}} = \frac{P_k(x)|x|^{-k}}{|x|^n} = \frac{P_k(x/|x|)}{|x|^n} = \frac{\Omega(x/|x|)}{|x|^n}\,,$$

where the function $\Omega(x) = P_k(x)|x|^{-k}$ satisfies

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} P_k(\theta)\, d\sigma(\theta) = 0\,. \tag{6.15}$$

Indeed, if ν is an outward normal vector to the unit sphere, then

$$\frac{\partial P_k}{\partial \nu} = \frac{d}{dt}\Big|_{t=1} P_k(tx) = kt^{k-1}P_k(x)\Big|_{t=1} = kP_k(x)$$

and hence Green's formula yields

$$k\int_{S^{n-1}} P_k(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} \frac{\partial P_k}{\partial \nu}(\theta)\, d\sigma(\theta) = \int_{B^n} \Delta P_k\, dx = 0\,.$$

Thus

$$W_\Omega[\varphi] = \text{p.v.} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{n+k}}\, \varphi(x)\, dx$$

is a well defined tempered distribution.

Our aim is to prove the following result.

Theorem 6.10. *If P_k is a homogeneous harmonic polynomial of degree $k \geq 1$, then*

$$\left(\text{p.v.} \frac{P_k(x)}{|x|^{n+k}}\right)^{\wedge}(\xi) = \gamma_k \frac{P_k(\xi)}{|\xi|^k}\,,$$

where

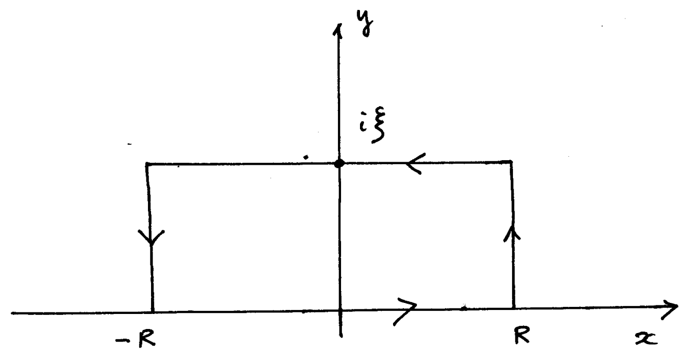
$$\gamma_k = (-i)^k \pi^{n/2} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)}\,.$$

Note that this result immediately implies Theorem 6.1.

Let us start with an alternative proof of the following fact (see Theorem 2.14).

Proposition 6.11. $\left(e^{-\pi x^2}\right)^{\wedge}(\xi) = e^{-\pi\xi^2}$.

Proof. The function $e^{-\pi z^2}$ is holomorphic and hence its integral along the following curve equals zero.



Letting $R \to \infty$ we obtain

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx\,.$$

The left hand side equals 1, so

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} e^{\pi\xi^2}\,dx\,.$$

Hence

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx = \left(e^{-\pi x^2}\right)^{\wedge}(\xi)\,.$$

The proof is complete. □

By the same argument involving the same contour integration, for any polynomial P we have

$$\int_{-\infty}^{\infty} P(x) e^{-\pi(x+i\xi)^2}\,dx = \int_{-\infty}^{\infty} P(x-i\xi) e^{-\pi x^2}\,dx\,. \tag{6.16}$$

If P is a polynomial in n variables, then (6.16) and the Fubini theorem yield

$$\int_{\mathbb{R}^n} P(x) e^{-\pi\sum_j (x_j+i\xi_j)^2}\,dx = \int_{\mathbb{R}^n} P(x-i\xi) e^{-\pi|x|^2}\,dx\,. \tag{6.17}$$

Theorem 6.12 (Bochner-Hecke)**.** *If $P_k(x)$ is a homogeneous harmonic polynomial of degree k, then*

$$\left(P_k(\cdot)e^{-\pi|\cdot|^2}\right)^{\wedge}(\xi) = (-i)^k P_k(\xi) e^{-\pi|\xi|^2}\,.$$

Proof. Applying the differential operator $P_k(D_\xi)$ to both sides of the identity

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi}\,dx = e^{-\pi|\xi|^2}$$

we see that

$$\int_{\mathbb{R}^n} P_k(x) e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} = Q(\xi) e^{-\pi|\xi|^2} \tag{6.18}$$

for some polynomial Q and it remains to prove that $Q(\xi) = P_k(-i\xi)$. Multiplying both sides of (6.18) by $e^{\pi|\xi|^2}$ and applying (6.17) we have

$$Q(\xi) = \int_{\mathbb{R}^n} P_k(x) e^{-\pi\sum_j (x_j+i\xi_j)^2}\,dx = \int_{\mathbb{R}^n} P_k(x-i\xi) e^{-\pi|x|^2}\,dx\,.$$

We can write

$$P_k(x-\eta) = \sum_\alpha \eta^\alpha P_\alpha(x)$$

and then

$$\int_{\mathbb{R}^n} P_k(x-\eta) e^{-\pi|x|^2}\,dx = \sum_\alpha \eta^\alpha \int_{\mathbb{R}^n} P_\alpha(x) e^{-\pi|x|^2}\,dx\,,$$

so clearly the integral is a polynomial is η. With this notation we obtain

$$Q(\xi)=\sum_\alpha (i\xi)^\alpha\int_{\mathbb{R}^n} P_\alpha(x)e^{-\pi|x|^2}\,dx$$

and hence

$$Q(\xi/i)=\sum_\alpha \xi^\alpha\int_{\mathbb{R}^n} P_\alpha(x)e^{-\pi|x|^2}\,dx=\int_{\mathbb{R}^n} P_k(x-\xi)e^{-\pi|x|^2}\,dx\,.$$

Since P_k is a harmonic function it has the mean value property

$$\int_{S^{n-1}} P_k(s\theta-\xi)\,d\sigma(\theta)=|S^{n-1}|P_k(-\xi)\,.$$

Thus integration in polar coordinates gives

$$\begin{aligned}
Q(\xi/i) &= \int_0^\infty s^{n-1}\left(\int_{S^{n-1}} P_k(s\theta-\xi)\,d\sigma(\theta)\right)e^{-\pi s^2}\,ds\\
&= P_k(-\xi)\int_0^\infty s^{n-1}|S^{n-1}|e^{-\pi s^2}\,ds\\
&= P_k(-\xi)\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx=P_k(-\xi)
\end{aligned}$$

and hence $Q(\xi)=P_k(-i\xi)$. The proof is complete. □

Using homogeneity of P_k and the second part of Theorem 2.7(e) one can easily deduce from the Bochner-Hecke formula the folloiwing result.

Corollary 6.13. *If P_k is a homogeneous harmonic polynomial of degree k, then for any $t>0$ we have*

$$\left(P_k(\cdot)e^{-\pi t|\cdot|^2}\right)^\wedge(\xi)=t^{-k-\frac{n}{2}}(-i)^kP_k(\xi)e^{-\pi|\xi|^2/t}\,.$$

We leave details as an easy exercise.

We will deduce Theorem 6.10 from the following result.

Theorem 6.14. *If P_k is a homogeneous harmonic polynomial of order k and $0<\alpha<n$, then*

$$\left(\frac{P_k(x)}{|x|^{k+n-\alpha}}\right)^\wedge(\xi)=\gamma_{k,\alpha}\,\frac{P_k(\xi)}{|\xi|^{k+\alpha}}\,,$$

where

$$\gamma_{k,\alpha}=(-i)^k\pi^{\frac{n}{2}-\alpha}\,\frac{\Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k+n-\alpha}{2}\right)}\,.$$

Remark. Note that the function

$$\frac{P_k(x)}{|x|^{k+n-\alpha}}$$

is a tempered L^1 function, so it defines a tempered distribution without necessity of taking the principal value of the integral.

Proof. For any $t > 0$ and $\varphi \in \mathcal{S}_n$ Corollary 6.13 gives

$$\int_{\mathbb{R}^n} P_k(x) e^{-\pi t|x|^2} \hat{\varphi}(x)\, dx = (-i)^k \int_{\mathbb{R}^n} P_k(x) e^{-\pi |x|^2/t} t^{-k-\frac{n}{2}} \varphi(x)\, dx\,.$$

Now we multiply both sides by

$$t^{\beta-1}, \quad \text{where} \quad \beta = \frac{k+n-\alpha}{2} > 0$$

and integrate with respect to $0 < t < \infty$. Since

$$\int_0^\infty e^{-\pi t|x|^2} t^{\beta-1}\, dt = (\pi|x|^2)^{-\beta}\Gamma(\beta)$$

the integral on the left hand side will be equal to

$$\text{(6.19)} \qquad \frac{\Gamma(\beta)}{\pi^\beta} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}}\, \hat{\varphi}(x)\, dx = \frac{\Gamma(\beta)}{\pi^\beta} \left(\frac{P_k(\cdot)}{|\cdot|^{k+n-\alpha}} \right)^{\wedge} [\varphi]\,.$$

Similarly

$$\begin{aligned} \int_0^\infty e^{-\pi|x|^2/t} t^{-k-\frac{n}{2}} t^{\beta-1}\, dt &= \int_0^\infty e^{-\pi|x|^2/t} t^{-\frac{k+\alpha}{2}-1}\, dt \\ &\overset{s=\pi|x|^2/t}{=} (\pi|x|^2)^{-\frac{k+\alpha}{2}} \int_0^\infty e^{-s} s^{\frac{k+\alpha}{2}-1}\, ds \\ &= (\pi|x|^2)^{-\frac{k+\alpha}{2}} \Gamma\left(\frac{k+\alpha}{2}\right)\,. \end{aligned}$$

Thus the integral on the right hand side equals

$$\text{(6.20)} \qquad (-i)^k \frac{\Gamma\left(\frac{k+\alpha}{2}\right)}{\pi^{\frac{k+\alpha}{2}}} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}}\, \varphi(x)\, dx\,.$$

Since integrals at (6.19) and (6.20) are equal one to another the theorem follows. $\square$

Proof of Theorem 6.10. For $\varphi \in \mathcal{S}_n$ we take the identity

$$\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}}\, \hat{\varphi}(x)\, dx = \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}}\, \varphi(x)\, dx$$

and let $\alpha \to 0$. The right hand side converges to

$$(-i)^k \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^k}\, \varphi(x)\, dx\,.$$

To compute the limit on the left hand side we observe that the integral of $P_k(x)|x|^{-(k+n-\alpha)}$ on the unit ball equals zero, see (6.15) and hence

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}}\,\hat{\varphi}(x)\,dx \\
&\quad = \int_{|x|\leq 1} \frac{P_k(x)}{|x|^{k+n-\alpha}}\,(\hat{\varphi}(x)-\hat{\varphi}(0))\,dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n-\alpha}}\,\hat{\varphi}(x)\,dx \\
&\quad \stackrel{\alpha\to 0^+}{\longrightarrow} \int_{|x|\leq 1} \frac{P_k(x)}{|x|^{k+n}}\,(\hat{\varphi}(x)-\hat{\varphi}(0))\,dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n}}\,\hat{\varphi}(x)\,dx \\
&\quad = \lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq 1} \frac{P_k(x)}{|x|^{k+n}}\,(\hat{\varphi}(x)-\hat{\varphi}(0))\,dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n}}\,\hat{\varphi}(x)\,dx \\
&\quad = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{P_k(x)}{|x|^{k+n}}\,\hat{\varphi}(x)\,dx \\
&\quad = \left(\text{p.v.}\,\frac{P_k(x)}{|x|^{k+n}}\right)^{\wedge}[\varphi]\,.
\end{aligned}
$$

Comparing the above limits yields the result. □

6.3. **How to differentiate functions.** We plan to generalize Proposition 6.9 to a class of more general functions. Let's start with the following elementary result.

Theorem 6.15. *Suppose that $K \in C^1(\mathbb{R}^n\setminus\{0\})$ is such that both K and $|\nabla K|$ have polynomial growth for $|x|\geq 1$ and there are constants $C,\alpha>0$ such that*

$$
|K(x)| \leq \frac{C}{|x|^{n-1-\alpha}} \quad \textit{for } 0<|x|<1,
$$

$$
|\nabla K(x)| \leq \frac{C}{|x|^{n-\alpha}} \quad \textit{for } 0<|x|<1.
$$

Then $K\in\mathcal{S}_n'$ and the distributions partial derivatives $\partial K/\partial x_j$, $1\leq j\leq n$, coincide with pointwise derivatives, i.e. for $\varphi\in\mathcal{S}_n$

$$
\frac{\partial K}{\partial x_j}[\varphi] := -\int_{\mathbb{R}^n} K(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx = \int_{\mathbb{R}^n}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx\,.
$$

Proof. Let $A(\varepsilon)=\{x:\,\varepsilon\leq|x|\leq\varepsilon^{-1}\}$. From (6.13) we have

$$
\begin{aligned}
\frac{\partial K}{\partial x_j}[\varphi] &= \lim_{\varepsilon\to 0} -\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}} K(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx \\
&= \lim_{\varepsilon\to 0}\left(\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx - \int_{\partial A(\varepsilon)} K(x)\varphi(x)\,\nu_j\,d\sigma(x)\right).
\end{aligned}
$$

Since

$$
\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx = \int_{\mathbb{R}^n}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx
$$

it remains to show that

$$\lim_{\varepsilon\to 0}\int_{\partial A(\varepsilon)} K(x)\varphi(x)\,\nu_j\,d\sigma = 0\,.$$

The integral on the outer sphere $|x| = \varepsilon^{-1}$ converges to 0 since K has polynomial growth and φ rapidly converges to 0 as $|x|\to\infty$ and on the inner sphere $|x| = \varepsilon$ we have

$$\left|\int_{|x|=\varepsilon} K(x)\varphi(x)\,\nu_j\,d\sigma\right| \le \frac{C}{\varepsilon^{n-1-\alpha}}\,\varepsilon^{n-1} = C\varepsilon^{\alpha}\to 0 \quad \text{as } \varepsilon\to 0.$$

The proof is complete. □

An interesting problem is the case $\alpha = 0$, i.e. when K and ∇K satisfy the estimates

$$|K(x)| \le \frac{C}{|x|^{n-1}}\,, \quad |\nabla K(x)| \le \frac{C}{|x|^{n}} \quad \text{for } x\neq 0. \tag{6.21}$$

One such situation was described in Proposition 6.9.

Here we make an additional assumption about K. We assume that $K\in C^1(\mathbb{R}^n\setminus\{0\})$ is homogenelus of degree $1-n$, i.e.

$$K(x) = \frac{K(x/|x|)}{|x|^{n-1}} \quad \text{for } x\neq 0.$$

Since K is bounded in $\{|x|\ge 1\}$ and integrable in $\{|x|\le 1\}$ we have $K\in\mathcal{S}'_n$ and there is no need to interpret K through the principal value of the integral. The first estimate at (6.21) is satisfied. To see that the second estimate if satisfied too we observe that ∇K is homogeneous of degree $-n$. Indeed, for $1\le j\le n$ and $t>0$

$$\frac{\partial K}{\partial x_j}(tx)t = \frac{\partial}{\partial x_j}\,(K(tx)) = t^{1-n}\frac{\partial}{\partial x_j}\,K(x)$$

and hence

$$(\nabla K)(tx) = t^{-n}\nabla K(x)\,.$$

Thus

$$\nabla K(x) = \frac{(\nabla K)(x/|x|)}{|x|^{n}}, \quad x\neq 0$$

from which the second estimate at (6.21) follows.

Observe that $\nabla K(x)$ is not integrable at any neighborhood of 0, but we may try to consider the principal value of $\nabla K(x)$, i.e. the principal value of each of the partial derivatives $\partial K/\partial x_j$. To do this we have to check if the condition (6.14) is satisfied.

Theorem 6.16. *Suppose that $K\in C^1(\mathbb{R}^n\setminus\{0\})$ is homogeneous of degree $1-n$. Then $\nabla K(x)$ is homogeneous of degree $-n$. Moreover*

$$\int_{S^{n-1}} \nabla K(\theta)\,d\sigma(\theta) = 0\,. \tag{6.22}$$

Hence the condition (6.14) is satisfied and thus

$$\text{p.v.}\, \nabla K(x) \in \mathcal{S}'_n$$

is a well defined tempered distribution, i.e. for each $1 \le j \le n$

$$\text{p.v.}\, \frac{\partial K}{\partial x_j}(x) \in \mathcal{S}'_n\,.$$

Finally the distributional gradient ∇K *satisfies*

$$\underbrace{\nabla K}_{\textit{dist.}} = c\delta_0 + \text{p.v.}\, \underbrace{\nabla K(x)}_{\textit{pointwise}}\,, \tag{6.23}$$

where

$$c = \int_{S^{n-1}} K(x)\frac{x}{|x|}\, d\sigma(x)\,.$$

In other words for $\varphi \in \mathcal{S}_n$ *and* $1 \le j \le n$ *we have*

$$\frac{\partial K}{\partial x_j}[\varphi] := -\int_{\mathbb{R}^n} K(x)\,\frac{\partial \varphi}{\partial x_j}(x)\, dx = c_j\varphi(0) + \lim_{\varepsilon\to 0}\int_{|x|\ge\varepsilon} \frac{\partial K}{\partial x_j}(x)\varphi(x)\, dx\,,$$

where

$$c_j = \int_{S^{n-1}} K(x)\,\frac{x_j}{|x|}\, d\sigma(x)\,.$$

Proof. We already checked that $\nabla K(x)$ is homogeneous of degree $-n$. For $r > 1$ let $A(1,r) = \{x :\, 1 \le |x| \le r\}$. From the integration by parts formula (6.12) we have

$$\begin{aligned}\int_{1\le|x|\le r} \nabla K(x)\, dx &= \int_{\partial A(1,r)} K(x)\,\vec{\nu}(x)\, d\sigma(x)\\ &= -\int_{|x|=1} K(x)\,\frac{x}{|x|}\, d\sigma(x) + \int_{|x|=r} K(x)\,\frac{x}{|x|}\, d\sigma(x) = 0\,.\end{aligned}$$

Indeed, the last two integrals are equal by a simple change of variables and homogeneity of K. Thus the integral on the left hand side equals 0 independently of r. Hence its derivative with respect to r is also equal zero.

$$0 = \frac{d}{dr}\Big|_{r=1^+}\int_{1\le|x|\le r} \nabla K(x)\, dx = \int_{|x|=1} \nabla K(\theta)\, d\sigma(\theta)\,.$$

This proves (6.22). Therefore p.v. $\nabla K(x) \in \mathcal{S}'_n$ is a well defined tempered distribution. We are left with the proof that the distributional gradient ∇K

satisfies (6.23). Let $\varphi \in \mathcal{S}_n$. We have

$$\begin{aligned}\nabla K[\varphi] &:= -\int_{\mathbb{R}^n} K(x)\nabla\varphi(x)\,dx \\ &= \lim_{\varepsilon\to 0}\lim_{R\to\infty} -\int_{\varepsilon\le|x|\le R} K(x)\nabla\varphi(x)\,dx \\ &= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\left(\int_{\varepsilon\le|x|\le R}\nabla K(x)\,\varphi(x)\,dx - \int_{\partial A(\varepsilon,R)} K(x)\varphi(x)\,\vec{\nu}(x)\,d\sigma(x)\right) \\ &= \lim_{\varepsilon\to 0}\left(\int_{|x|\ge\varepsilon}\nabla K(x)\,\varphi(x)\,dx + \int_{|x|=\varepsilon} K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x)\right).\end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x) = \varphi(0)\int_{|x|=1} K(x)\,\frac{x}{|x|}\,d\sigma(x)\,.$$

Let

$$c = \int_{|x|=1} K(x)\,\frac{x}{|x|}\,d\sigma(x) = \int_{|x|=\varepsilon} K(x)\,\frac{x}{|x|}\,d\sigma(x)\,.$$

The last equality follows from a simple change of variables and homogeneity of K. We have

$$\begin{aligned}&\int_{|x|=\varepsilon} K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x) \\ &\quad= c\varphi(0) + \int_{|x|=\varepsilon} K(x)\big(\varphi(x)-\varphi(0)\big)\,\frac{x}{|x|}\,d\sigma(x) \\ &\quad\to c\varphi(0)\end{aligned} \tag{6.24}$$

as $\varepsilon\to 0$. Indeed, for $|x|=\varepsilon$

$$\left|K(x)\big(\varphi(x)-\varphi(0)\big)\,\frac{x}{|x|}\right| \le C\varepsilon^{1-n}\varepsilon = C\varepsilon^{2-n}\,.$$

Since the surface area of the sphere $\{|x|=\varepsilon\}$ is $n\omega_n\varepsilon^{n-1}$, the integral on the right hand side of (6.24) converges to 0 with $\varepsilon\to 0$. □

6.4. **Integral representations of functions.** A straightforward application of the above theorem gives a well known formula for the fundamental solution to the Laplace equation.

Theorem 6.17. *For $n\ge 2$ we have*

$$\Delta\Phi = \delta_0\,,$$

where

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n=2, \\ -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{if } n\ge 3.\end{cases}$$

Proof. We will prove the theorem for $n \geq 3$, but a similar argument works for $n = 2$. According to Theorem 6.15

$$\nabla \Phi = \frac{1}{n\omega_n} \frac{x}{|x|^n} \tag{6.25}$$

in the sense of distributions.[32] Note that the function $\Phi(x)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ and hence

$$0 = \Delta\Phi(x) = \operatorname{div} \nabla\Phi(x) = \frac{1}{n\omega_n} \sum_{j=1}^n \frac{\partial}{\partial x_j} \frac{x_j}{|x|^n} \quad \text{for } x \neq 0.$$

Now Theorem 6.16 gives a formula for the distributional Laplacean

$$\Delta\Phi = \operatorname{div} \nabla\Phi = c\delta_0 + \frac{1}{n\omega_n} \text{ p.v. } \underbrace{\sum_{j=1}^n \frac{\partial}{\partial x_j} \frac{x_j}{|x|^n}}_{0} = c\delta_0 \,,$$

where

$$c = \sum_{j=1}^n \int_{|x|=1} \frac{1}{n\omega_n} \frac{x_j}{|x|^n} \frac{x_j}{|x|} \, d\sigma(x) = \frac{1}{n\omega_n} \int_{|x|=1} d\sigma(x) = 1 \,.$$

The proof is complete. □

For $\varphi \in \mathcal{S}_n$ let $u(x) = (\Phi * \varphi)(x)$. Then[33] $u \in C^\infty(\mathbb{R}^n)$ and

$$\Delta u(x) = \Delta(\Phi * \varphi)(x) = \big((\Delta\Phi) * \varphi\big)(x) = (\delta_0 * \varphi)(x) = \varphi(x) \,.$$

Hence convolution with the fundamental solution of the Laplace operator provides an explicit solution to the Poisson equation

$$\Delta u = \varphi \,.$$

This explains the importance of the fundamental solution in partial differential equations.

Observe that the above calculation gives also

$$\begin{aligned}
\varphi(x) &= \Delta(\Phi * \varphi)(x) \\
&= \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} (\Phi * \varphi)(x) \\
&= \sum_{j=1}^n \left(\frac{\partial \Phi}{\partial x_j} * \frac{\partial \varphi}{\partial x_j} \right)(x) \\
&= \int_{\mathbb{R}^n} \nabla\Phi(x-y) \cdot \nabla\varphi(y) \, dy \\
&= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x-y) \cdot \nabla\varphi(y)}{|x-y|^n} \, dy
\end{aligned}$$

[32] Formula (6.25) is also true for $n = 2$.

[33] As a convolution of $\Phi \in \mathcal{S}_n'$ with $\varphi \in \mathcal{S}_n$.

for every $x \in \mathbb{R}^n$. In the last equality we employed (6.25). Thus we proved

Theorem 6.18. *For $\varphi \in \mathcal{S}_n$, $n \geq 2$ we have*

$$\varphi(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x-y)\cdot\nabla\varphi(y)}{|x-y|^n}\, dy \quad \textit{for all } x \in \mathbb{R}^n.$$

From this theorem we can conclude a similar result for higher order derivatives.

Theorem 6.19. *For $\varphi \in \mathcal{S}_n$, $n \geq 2$ and $m \geq 1$ we have*

$$\varphi(x) = \frac{m}{n\omega_n} \int_{\mathbb{R}^n} \sum_{|\alpha|=m} \frac{D^\alpha\varphi(y)}{\alpha!} \frac{(x-y)^\alpha}{|x-y|^n}\, dy \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. Fix $x \in \mathbb{R}^n$ and define

$$\psi(y) = \sum_{|\beta|\leq m-1} D^\beta\varphi(y)\, \frac{(x-y)^\beta}{\beta!}\,.$$

Then $\psi(x) = \varphi(x)$ and[34]

$$\frac{\partial\psi}{\partial y_j}(y) = \sum_{|\beta|=m-1} D^{\beta+\delta_j}\varphi(y)\, \frac{(x-y)^\beta}{\beta!}$$

where[35] $\delta_j = (0, \ldots, 1, \ldots, 0)$. Hence Theorem 6.18 applied to ψ yields

$$\begin{aligned}
\varphi(x) &= \psi(x) \\
&= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial\psi}{\partial y_j}(y)\, \frac{x_j - y_j}{|x-y|^n}\, dy \\
&= \frac{1}{n\omega_n} \sum_{j=1}^n \sum_{|\beta|=m-1} \int_{\mathbb{R}^n} D^{\beta+\delta_j}\varphi(y)\, \frac{(x-y)^\beta}{\beta!}\, \frac{x_j-y_j}{|x-y|^n}\, dy \\
&= \frac{m}{n\omega_n} \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \frac{D^\alpha\varphi(y)}{\alpha!}\, \frac{(x-y)^\alpha}{|x-y|^n}\, dy\,,
\end{aligned}$$

because for α with $|\alpha| = m$

$$\sum_{j,\beta:\, \beta+\delta_j=\alpha} \frac{1}{\beta!} = \frac{m}{\alpha!}\,.$$

The proof is complete. □

For $0 < \alpha < n$ and $n \geq 2$ we define the *Riesz potentials* by

$$(I_\alpha f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy\,,$$

[34] We compute $\partial\psi/\partial y_j$ using the Leibniz rule and observe that the lower order terms cancel out.

[35] 1 on jth coordinate, 0 otherwise.

where
$$\gamma(\alpha)=\frac{\pi^{\frac{n}{2}}\,2^\alpha\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\,.$$
In particular, when $n\geq 3$, I_2f is the convolution with the fundamental solution to the Laplace operator taken with the minus sign, so for $\varphi\in\mathcal{S}_n$
$$-\Delta(I_2\varphi)(x)=\varphi(x)\,.$$
If $\alpha=1$, then, up to a constant, I_1f is a convolution with the distribution $|x|^{1-n}\in\mathcal{S}_n'$.
$$(I_1f)(x)=\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\,\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-1}}\,dy\,.$$
Hence for $\varphi\in\mathcal{S}_n$, $I_1\varphi\in C^\infty$ and Proposition 6.9 gives
$$\frac{\partial}{\partial x_j}(I_1\varphi)(x)=(1-n)\,\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\,\pi^{\frac{n+1}{2}}}\,\text{p.v.}\int_{\mathbb{R}^n}\frac{x_j-y_j}{|x-y|^{n+1}}\,\varphi(y)\,dy=-R_j\varphi(x).$$
Thus the Riesz operators appear naturally as derivatives of the integral operator $I_1\varphi$. We proved

Proposition 6.20. *If $n\geq 2$ and $\varphi\in\mathcal{S}_n$, then for $1\leq j\leq n$*
$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\,\pi^{\frac{n+1}{2}}}\,\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n}\frac{\varphi(y)}{|x-y|^{n-1}}\,dy=-R_j\varphi(x)\,.$$

7. Singular integrals I

For $\Omega\in L^1(S^{n-1})$ with
$$\int_{S^{n-1}}\Omega(\theta)\,d\sigma(\theta)=0 \tag{7.1}$$
we define
$$K_\Omega(x)=\frac{\Omega(x/|x|)}{|x|^n}\,,\quad x\neq 0\,.$$
Then for $\varphi\in\mathcal{S}_n$ we define the tempered distribution
$$\begin{aligned}W_\Omega[\varphi]&=\text{p.v.}\int_{\mathbb{R}^n}K_\Omega(x)\varphi(x)\,dx\\&=\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}K_\Omega(x)\varphi(x)\,dx\end{aligned}$$
As in the case of Theorem 5.1 one can prove that for $\varphi\in\mathcal{S}_n$ the limit exists and defines $W_\Omega\in\mathcal{S}_n'$. Note that the condition (7.1) plays an essential role in the proof.

The following result generalizes Theorem 6.1.

Theorem 7.1. *Let $n \geq 2$ and $\Omega \in L^1(S^{n-1})$ be such that*

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0\,.$$

Then the Fourier transform of the distribution W_Ω is a finite a.e. function given by the formula

$$\begin{aligned} \widehat{W_\Omega}(\xi) &= \int_{S^{n-1}} \Omega(\theta) \left(\log \frac{1}{|\xi \cdot \theta|} - \frac{i\pi}{2} \operatorname{sgn}(\xi \cdot \theta) \right) d\sigma(\theta) \\ &= \int_{S^{n-1}} \Omega(\theta) \left(\log \frac{1}{|\xi' \cdot \theta|} - \frac{i\pi}{2} \operatorname{sgn}(\xi' \cdot \theta) \right) d\sigma(\theta)\,, \end{aligned} \tag{7.2}$$

where $\xi' = \xi/|\xi|$.

Before we prove the theorem we start with some auxiliary results.

Lemma 7.2. *Let K be a function of one variable, then for $n \geq 2$ we have*

$$\int_{S^{n-1}} K(x \cdot \theta)\, d\sigma(\theta) = (n-1)\omega_{n-1} \int_{-1}^{1} K(s|x|)(1-s^2)^{\frac{n-3}{2}}\, ds$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

This result follows from arguments similar to those used to establish (6.7); we leave details to the reader as an exercise.

Lemma 7.3. *If $h : [0, \infty) \to \mathbb{R}$ is continuous, bounded and the improper integral*

$$\int_1^\infty \frac{h(s)}{s}\, ds$$

converges, then for $\mu > \lambda > 0$ and $N > \varepsilon > 0$ we have

$$\left| \int_\varepsilon^N \frac{h(\lambda s) - h(\mu s)}{s}\, ds \right| \leq 2\|h\|_\infty \log\left(\frac{\mu}{\lambda}\right)\,. \tag{7.3}$$

Moreover

$$\lim_{\substack{N\to\infty \\ \varepsilon \to 0}} \int_\varepsilon^N \frac{h(\lambda s) - h(\mu s)}{s}\, ds = h(0) \log\left(\frac{\mu}{\lambda}\right)\,. \tag{7.4}$$

Proof. We have

$$\begin{aligned} \int_\varepsilon^N \frac{h(\lambda s) - h(\mu s)}{s}\, ds &= \int_{\lambda\varepsilon}^{\lambda N} \frac{h(s)}{s}\, ds - \int_{\mu\varepsilon}^{\mu N} \frac{h(s)}{s}\, ds \\ &= \int_{\lambda\varepsilon}^{\mu\varepsilon} \frac{h(s)}{s}\, ds - \int_{\lambda N}^{\mu N} \frac{h(s)}{s}\, ds\,. \end{aligned}$$

Estimating the absolute value of the last two integrals gives (7.3). Since

$$\int_{\lambda\varepsilon}^{\mu\varepsilon} \frac{h(s)}{s}\, ds \to h(0) \log\left(\frac{\mu}{\lambda}\right) \quad \text{as } \varepsilon \to 0$$

and

$$\int_{\lambda N}^{\mu N} \frac{h(s)}{s}\, ds \to 0 \quad \text{as } N \to \infty$$

(7.4) follows. □

Corollary 7.4. *For $a \neq 0$ we have*

$$\lim_{\substack{N\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^N \frac{e^{-isa} - \cos s}{s}\, ds = \log\frac{1}{|a|} - i\frac{\pi}{2}\operatorname{sgn} a$$

and the integral is bounded by a constant independent of ε and N.

Proof. We have

$$\begin{aligned}\int_\varepsilon^N \frac{e^{-isa} - \cos s}{s}\, ds &= \int_\varepsilon^N \frac{\cos(sa) - \cos s}{s}\, ds - i\int_\varepsilon^N \frac{\sin(sa)}{s}\, ds\\ &= \int_\varepsilon^N \frac{\cos(s|a|) - \cos s}{s} - i\operatorname{sgn}(a)\int_\varepsilon^N \frac{\sin(s|a|)}{s}\, ds\end{aligned}$$

and the result follows from Lemma 7.3. □

Proof of Theorem 7.1. First observe that the last equality in (7.2) follows from

$$\log\frac{1}{|\xi\cdot\theta|} = \log\frac{1}{|\xi|} + \log\frac{1}{|\xi'\cdot\theta|}\,,$$

and the fact that

$$\int_{S^{n-1}} \Omega(\theta)\log\frac{1}{|\xi|}\, d\sigma(\theta) = 0\,.$$

Let $F(\xi)$ be the function defined by the integral in the second line of (7.2). We will show first that F is finite a.e. and that it actually defines a tempered distribution. Note that

$$\int_{S^{n-1}} \Omega(\theta)\,\frac{i\pi}{2}\operatorname{sgn}(\xi'\cdot\theta)\, d\sigma(\theta)$$

is a bounded function of ξ, so this component of $F(\xi)$ does not cause any troubles and hence we only need to estimate

$$G(\xi) = \int_{S^{n-1}} \Omega(\theta)\log\frac{1}{|\xi'\cdot\theta|}\, d\sigma(\theta)\,.$$

We need to show that the integral is finite for a.e. ξ and that

$$G[\varphi] = \int_{\mathbb{R}^n} G(\xi)\varphi(\xi)\, d\xi$$

is a tempered distribution, $\varphi \in \mathcal{S}_n$. To this end it suffices to show that the function $G(\xi)\varphi(\xi)$ is integrable along with suitable estimates for its integral.

We have

$$\begin{aligned}|G[\varphi]| &\le \int_{\mathbb{R}^n} |\varphi(\xi)| \int_{S^{n-1}} |\Omega(\theta)| \log\frac{1}{|\xi'\cdot\theta|}\, d\sigma(\theta)\, d\xi \\ &= \int_{S^{n-1}} |\Omega(\theta)| \int_0^\infty s^{n-1} \int_{S^{n-1}} |\varphi(s\xi')| \log\frac{1}{|\xi'\cdot\theta|}\, d\sigma(\xi')\, ds\, d\sigma(\theta) \\ &= \heartsuit\,.\end{aligned}$$

Lemma 7.2 gives

$$\begin{aligned}(7.5)\qquad &\int_{S^{n-1}} \log\frac{1}{|\xi'\cdot\theta|}\, d\sigma(\xi') \\ &= (n-1)\omega_{n-1} \int_{-1}^1 \left(\log\frac{1}{|s|}\right) (1-s^2)^{\frac{n-3}{2}}\, ds = c_n < \infty\end{aligned}$$

and hence

$$\begin{aligned}\heartsuit &\le c_n \int_{S^{n-1}} |\Omega(\theta)| \int_0^\infty s^{n-1} \sup_{|\xi|=s} |\varphi(\xi)|\, ds\, d\sigma(\theta) \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \left(\|\varphi\|_\infty + \sup_{\xi\in\mathbb{R}^n} |\xi|^{n+1}|\varphi(\xi)|\right).\end{aligned}$$

This proves that G is finite a.e. and defines a tempered distribution, so does F.

Now we are ready to prove formula (7.2). Let $\xi' = \xi/|\xi|$. We have

$$\begin{aligned}&\widehat{W_\Omega}[\varphi] = W_\Omega[\hat{\varphi}] \\ &= \lim_{\varepsilon\to 0} \int_{|x|\ge\varepsilon} \frac{\Omega(x/|x|)}{|x|^n}\, \hat{\varphi}(x)\, dx \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{\varepsilon\le|x|\le N} \frac{\Omega(x/|x|)}{|x|^n}\, e^{-2\pi i x\cdot\xi}\, dx\, d\xi \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N e^{-2\pi i s\theta\cdot\xi}\, \frac{ds}{s}\, d\sigma(\theta)\, d\xi \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N \left(e^{-2\pi i s\theta\cdot\xi} - \cos(2\pi s|\xi|)\right) \frac{ds}{s}\, d\sigma(\theta)\, d\xi \\ &= \diamondsuit\,.\end{aligned}$$

The last equality follows from the fact that the integral of Ω over the sphere vanishes. We have

$$\begin{aligned}\int_\varepsilon^N \left(e^{-2\pi i s\theta\cdot\xi} - \cos(2\pi s|\xi|)\right) \frac{ds}{s} &= \int_{2\pi|\xi|\varepsilon}^{2\pi|\xi|N} \frac{e^{-is\theta\cdot\xi'} - \cos s}{s}\, ds \\ &\longrightarrow \log\frac{1}{|\theta\cdot\xi'|} - i\frac{\pi}{2}\,\mathrm{sgn}\,(\theta\cdot\xi')\end{aligned}$$

by Corollary 7.4 and hence the dominated convergence theorem gives

$$\diamondsuit \quad = \quad \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \left(\log \frac{1}{|\theta \cdot \xi'|} - i\frac{\pi}{2} \operatorname{sgn}(\theta \cdot \xi') \right) d\sigma(\theta)\, d\xi \,.$$

The proof is complete. □

If Ω is an odd function, i.e. $\Omega(\theta) = -\Omega(-\theta)$, then the integral of Ω against $\log(1/|\xi \cdot \theta|)$ vanishes and hence

$$\widehat{W_\Omega}(\xi) = -\frac{i\pi}{2} \int_{S^{n-1}} \Omega(\theta) \operatorname{sgn}(\xi' \cdot \theta)\, d\sigma(\theta)\,.$$

In particular the Fourier transform $\widehat{W_\Omega}$ is bounded. More generally any function Ω on S^{n-1} can be decomposed into its even and odd parts

$$\Omega_e(\theta) = \frac{1}{2}(\Omega(\theta) + \Omega(-\theta)), \qquad \Omega_o(\theta) = \frac{1}{2}(\Omega(\theta) - \Omega(-\theta))\,.$$

Corollary 7.5. *Let $\Omega \in L^1(S^{n-1})$ be such that*

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0\,.$$

If $\Omega_o \in L^1(S^{n-1})$ and $\Omega_e \in L^q(S^{n-1})$ for some $q > 1$, then the Fourier transform of W_Ω is a bounded function.

Proof. A calculation similar to (7.5) implies that $\log(1/|\xi' \cdot \theta|)$ is integrable over S^{n-1} with any positive exponent. In particular it belongs to $L^{q'}(S^{n-1})$ and hence

$$\left| \int_{S^{n-1}} \Omega_e(\theta) \left(\log \frac{1}{|\xi' \cdot \theta|} - \frac{i\pi}{2} \operatorname{sgn}(\xi' \cdot \theta) \right) d\sigma(\theta) \right| \leq C \|\Omega_e\|_{L^q(S^{n-1})}\,,$$

with a constant C independent of ξ. Now the formula (7.2) yields

$$\begin{aligned} \widehat{W_\Omega}(\xi) \quad &= \quad -\frac{i\pi}{2} \int_{S^{n-1}} \Omega_o(\theta) \operatorname{sgn}(\xi' \cdot \theta)\, d\sigma(\theta) \\ &+ \quad \int_{S^{n-1}} \Omega_e(\theta) \left(\log \frac{1}{|\xi' \cdot \theta|} - \frac{i\pi}{2} \operatorname{sgn}(\xi' \cdot \theta) \right) d\sigma(\theta) \end{aligned}$$

from which we have

$$\|\widehat{W_\Omega}\|_\infty \leq C(\|\Omega_o\|_{L^1(S^{n-1})} + \|\Omega_e\|_{L^q(S^{n-1})})\,.$$

The proof is complete. □

Convolutions with W_Ω, i.e. operators of the form

$$T_\Omega f(x) = (W_\Omega * f)(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy$$

are called *singular integrals*. Examples include the Hilbert transform and the Riesz transforms.

If Ω satisfies assumptions of Corollary 7.5, then the singular integral T_Ω uniquely extends from $\mathcal{S}_n$ to a bounded operator on L^2. However we are interested to see whether there is a more direct formula to define $T_\Omega f(x)$ when $f \in L^2$ without using the density argument. Moreover we want to know if the operator is bounded in L^p for $p \neq 2$. More precisely we want to know if Corollary 5.17 proved for the Hilbert transform generalizes to the more general class of singular integrals T_Ω.

Definition. For a function $\Omega \in L^1(S^{n-1})$ with vanishing integral and $f \in L^p(S^{n-1})$, $1 \leq p < \infty$ we define the *truncated singular integral*

$$T_\Omega^{(\varepsilon,N)} f(x) = \int_{\varepsilon \leq |y| \leq N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy, \quad 0 < \varepsilon < N\,.$$

Note that

$$\|T_\Omega^{(\varepsilon,N)} f\|_p \leq \|\Omega\|_{L^1(S^{n-1})} \log(N/\varepsilon) \|f\|_p$$

and hence $T_\Omega^{(\varepsilon,N)} f$ is well defined. The *maximal singular integral* is defined by

$$T_\Omega^* f(x) = \sup_{0<\varepsilon<N<\infty} |T_\Omega^{(\varepsilon,N)} f(x)|\,.$$

In an important case when Ω is bounded,[36] $\Omega(y/|y|)/|y|^n$ belongs to $L^q(\{|y| \geq \varepsilon\})$ for any $1 < q \leq \infty$ and hence for $f \in L^p$, $1 \leq p < \infty$ the integral

$$T_\Omega^\varepsilon f(x) = \int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy$$

is finite. Since

$$T_\Omega^\varepsilon f(x) = \lim_{N\to\infty} T_\Omega^{(\varepsilon,N)} f(x)$$

and

$$T_\Omega^{(\varepsilon,N)} f(x) = T_\Omega^\varepsilon f(x) - T_\Omega^N f(x)$$

we have

$$\frac{1}{2} T_\Omega^* f(x) \leq \sup_{\varepsilon>0} |T_\Omega^\varepsilon f(x)| \leq T_\Omega^* f(x) \tag{7.6}$$

and hence in the case in which Ω is bounded we could define the maximal singular operator as the left hand side of (7.6).

7.1. The method of rotations.

The following result proves boundedness of singular integrals in L^p, $1 < p < \infty$. The proof is based on the so called *method of rotations.*

Theorem 7.6. *If $\Omega \in L^1(S^{n-1})$ is an odd function, then T_Ω^* is of strong type (p,p) for all $1 < p < \infty$. In particular T_Ω uniquely extends from $\mathcal{S}_n$ to a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

[36] This covers the case of the Hilbert transform, the Riesz transform and more generally the case of transforms with the kernel $P_k(x)/|x|^{n+k}$.

Proof. The idea is to show that the singular integral $T_\Omega f$ is an average over all possible directions in $\mathbb{R}^n$ of one dimensional directional Hilbert transforms and then the result will follow from the corresponding results about boundedness of one dimensional Hilbert transform.

If e_1 is the direction of the first coordinate, then the operator

$$H_{e_1} f(x) = \frac{1}{\pi}\,\text{p.v.} \int_{-\infty}^{\infty} \frac{f(x-te_1)}{t}\,dt$$

is bounded in $L^p(\mathbb{R}^n)$, $1<p<\infty$. Indeed, for a.e. $(x_2,\ldots,x_n)\in\mathbb{R}^{n-1}$ the function $z\mapsto f(z,x_2,\ldots,x_n)$ belongs to $L^p(\mathbb{R})$ and hence the one dimensional truncated Hilbert transform applied to the first coordinate

$$H^\varepsilon f(z,x_2,\ldots,x_n) = \frac{1}{\pi}\int_{|t|\geq\varepsilon} \frac{f(z-t,x_2,\ldots,x_n)}{t}\,dt$$

converges a.e. with respect to z and in $L^p(\mathbb{R})$ to the Hilbert transform applied to the first coordinate, which is $H_{e_1}f(z,x_2,\ldots,x_n)$, see Corollary 5.17. Hence the Fubini theorem yields

$$\begin{aligned}
\|H_{e_1}f\|_p &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} |H_{e_1}f(z,x_2,\ldots,z_n)|^p\,dz\,dx_2,\ldots\,dx_n \\
&\leq C(p)^p\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} |f(z,x_2,\ldots,x_n)|^p\,dz\,dx_2\,\ldots\,dx_n \\
&= C(p)^p\|f\|_p^p\,,
\end{aligned}$$

where $C(p)$ stands for the norm of the Hilbert transform in $L^p(\mathbb{R})$.

Now for a direction $\theta\in S^{n-1}$ we define the *directional Hilbert transform* as

$$H_\theta f(x) = \frac{1}{\pi}\,\text{p.v.} \int_{-\infty}^{\infty} \frac{f(x-t\theta)}{t}\,dt\,.$$

The directional Hilbert transform commutes with the orthogonal transformations, i.e. for $\rho\in O(n)$

$$H_{\rho(e_1)}f(x) = H_{e_1}(f\circ\rho)(\rho^{-1}x)\,.$$

This identity follows immediately from the definition of $H_\theta f$. Hence H_θ is bounded in L^p with the norm independent of θ

$$\|H_\theta f\|_p \leq C(p)\|f\|_p\,.$$

Similarly for $f\in L^p(\mathbb{R}^n)$, $1\leq p<\infty$ we define

$$H_\theta^{(\varepsilon,N)}f(x) = \frac{1}{\pi}\int_{\varepsilon\leq|t|\leq N} \frac{f(x-t\theta)}{t}\,dt\,,$$

$$H_\theta^* f(x) = \sup_{0<\varepsilon<N<\infty} |H_\theta^{(\varepsilon,N)}f(x)|\,. \tag{7.7}$$

The two operators also commute with the orthogonal transformations and hence boundedness of H^*_θ in L^p, $1<p<\infty$ will follow from the boundedness if $H^*_{e_1}$.

Observe that for the one dimensional Hilbert transform

$$|H^{(\varepsilon,N)}g(x)| \le |H^N g(x)| + |H^\varepsilon g(x)|$$

and hence

$$\sup_{0<\varepsilon<N} |H^{(\varepsilon,N)}g(x)| \le 2H^*g(x)$$

where[37]

$$H^*g(x) = \sup_{\varepsilon>0} |H^\varepsilon g(x)|\,.$$

Thus boundedness of $H^*_{e_1}$ in L^p, $1<p<\infty$ follows from Corollary 5.16 and the Fubini theorem. Then also H^*_θ is bounded in L^p, $1<p<\infty$ with a constant independent of θ

$$\|H^*_\theta f\|_p \le 2A(p)\|f\|_p\,,$$

where $A(p)$ is the L^p norm of the operator $H^*: L^p(\mathbb{R}) \to L^p(\mathbb{R})$.

Now we will show how to represent the singular integral $T_\Omega f$ as an average of directional Hilbert transforms. For $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ we have

$$\begin{aligned}\int_{\varepsilon\le|y|\le N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy &= +\int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N \frac{f(x-t\theta)}{t}\,dt\,d\sigma(\theta)\\ &= -\int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N \frac{f(x+t\theta)}{t}\,dt\,d\sigma(\theta)\,.\end{aligned}$$

The first equality is just the representation of the integral in polar coordinates, while the second one follows from the change of variables $\theta \mapsto -\theta$ and the fact that $\Omega(-\theta) = -\Omega(\theta)$. Hence

$$\begin{aligned}&\int_{\varepsilon\le|y|\le N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy\\ &\quad= \frac{1}{2}\int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N \frac{f(x-t\theta)-f(x+t\theta)}{t}\,dt\,d\sigma(\theta)\\ &\quad= \frac{\pi}{2}\int_{S^{n-1}} \Omega(\theta) H_\theta^{(\varepsilon,N)} f(x)\,d\sigma(\theta)\,.\end{aligned} \tag{7.8}$$

Thus

$$T^*_\Omega f(x) \le \frac{\pi}{2}\int_{S^{n-1}} |\Omega(\theta)| H^*_\theta f(x)\,d\sigma(\theta)$$

and for $1<p<\infty$ the Minkowski integral inequality yields

$$\|T^*_\Omega f\|_p \le \pi A(p) \|\Omega\|_{L^1(S^{n-1})} \|f\|_p\,.$$

[37]Observe that the definition of H^*g is not consistent with the definition of $H^*_\theta f$. In the first definition we take supremum over $\varepsilon>0$, while in the second one supremum over $0<\varepsilon<N$. However, we need the maximal function H^*g, because we want to apply Corollary 5.16.

The proof is complete. □

Note that for $\varphi \in \mathcal{S}_n$, $H_\theta^{(\varepsilon,N)}\varphi(x)$ is bounded by a constant independent of ε, N and θ, so passing to the limit in (7.8) yields

$$T_\Omega\varphi(x) = \frac{\pi}{2}\int_{S^{n-1}} \Omega(\theta) H_\theta\varphi(x)\, d\sigma(\theta)\,.$$

Corollary 7.7. *If $\Omega \in L^1(S^{n-1})$ is an odd function, and $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then $T_\Omega^{(\varepsilon,N)} f \to T_\Omega f$ a.e. and in L^p as $\varepsilon \to 0$ and $N \to \infty$. If in addition Ω is a bounded function, then $T_\Omega^\varepsilon f \to T_\Omega f$ a.e. and in L^p as $\varepsilon \to 0$.*

The proof is almost the same as that for Corollary 5.17; we leave details to the reader.

One can actually prove a stronger result.

Theorem 7.8. *If $\Omega \in L^1(S^{n-1})$ has vanishing integral and $\Omega_e \in L^q(S^{n-1})$ for some $q \geq 1$, then for any $1 < p < \infty$ there is a constant $C > 0$ such that*

$$\|T_\Omega\varphi\|_p \leq C\|\varphi\|_p, \quad \varphi \in \mathcal{S}_n\,.$$

Hence T_Ω uniquely extends to a bounded operator on L^p. Moreover for $f \in L^p(\mathbb{R}^n)$, $T_\Omega^{(\varepsilon,N)} f \to T_\Omega f$ a.e. and in L^p as $\varepsilon \to 0$, $N \to \infty$.

The estimate for the odd part of Ω was done above, so we can assume that Ω is even, i.e. $\Omega = \Omega_e \in L^q$. In this case the method of rotations cannot be applied directly. However since

$$\sum_{j=1}^n R_j^2 = -I$$

we have, at least formally,

$$T_\Omega f = -\sum_{j=1}^n R_j^2(T_\Omega f) = -\sum_{j=1}^n R_j(R_j T_\Omega f)\,.$$

The operator $R_j T_\Omega$ is odd as a composition of an even and an odd operator, so we can apply the method of rotations to estimate it. However, the details are not as easy as they seem and we will prove it.

8. Singular integrals II

So far we investigated boundedness of singular integrals for $1 < p < \infty$, without investigating the case $p = 1$, but it turns out that a more powerful method is based on the weak $(1,1)$ estimates and the Marcinkiewicz interpolation theorem. First we will consider the simplest case of the Hilbert transform.

8.1. **Hilbert transform again.** We proved that the Hilbert transform is bounded in L^p, $1 < p < \infty$ by two different methods. We will add one more method now. Namely we will show that the Hilbert transform is of weak type $(1,1)$. This fact, the Marcinkiewicz interpolation theorem and a duality argument will easily imply boundedness of the Hilbert transform in L^p for all $1 < p < \infty$.

Theorem 8.1 (Kolmogorov)**.** *The Hilbert transform in of weak type* $(1,1)$. *More precisely for* $f \in L^1 \cap L^2(\mathbb{R})$ *we have*

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| \leq \frac{C}{t} \|f\|_1 .$$

In the proof we will use boundedness of the Hilbert transform in L^2, Theorem 5.3, and we will not refer to any result proved after Theorem 5.3. In particular the Hilbert transform in Kolmogorov's theorem is defined on L^2 as an extension from $\mathcal{S}(\mathbb{R})$. As we already mentioned Kolmogorov's theorem implies

Corollary 8.2 (Riesz)**.** *The Hilbert transform is bounded in* L^p, $1 < p < \infty$,

$$\|Hf\|_p \leq C\|f\|_p \quad \text{for } f \in \mathcal{S}(\mathbb{R}).$$

Proof. Since the Hilbert transform is bounded in L^2 (Theorem 5.3) and of weak type $(1,1)$, the Marcinkiewicz interpolation theorem implies that for all $1 < p < 2$,

$$\|Hf\|_p \leq C_p\|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}) .$$

Now the duality argument, Theorem 4.6, yields that for all $2 < p < \infty$ we also have

$$\|Hf\|_p \leq C_p\|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}) .$$

More directly, the duality argument goes as follows. For $f, g \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$ we have

$$\int_{\mathbb{R}} H^\varepsilon f \cdot g = \int_{\mathbb{R}} f \cdot H^\varepsilon g .$$

Letting $\varepsilon \to 0$ and using the fact that $H^\varepsilon u \to Hu$ in L^2 for $u \in L^2$ (Theorem 5.3) we have

$$\int_{\mathbb{R}} Hf \cdot g = \int_{\mathbb{R}} f \cdot g .$$

Now if $2 < p < \infty$, then

$$\begin{aligned} \|Hf\|_p &= \sup\left\{\left|\int_{\mathbb{R}} Hf \cdot g\right| : \|g\|_{p'} \leq 1\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}} f \cdot Hg\right| : \|g\|_{p'} \leq 1\right\} \\ &\leq \|f\|_p \sup\{\|Hg\|_{p'} : \|g\|_{p'} \leq 1\} \\ &\leq C_{p'}\|f\|_p . \end{aligned}$$

The proof is complete. □

Proof of Theorem 8.1. We can assume that $f \geq 0$. Indeed, every complex-valued function is a linear combination of nonnegative functions: positive and negative parts of real and imaginary part of the function.

We fix $t > 0$ and apply the Calderón-Zygmund decomposition (Theorem 1.13 and Corollary 1.14) to f and $\alpha = t$. We obtain non-overlapping intervals $\{I_j\}$ such that

$$f(x) \leq t \quad \text{for a.e. } x \notin \Omega = \bigcup_j I_j,$$

$$|\Omega| \leq \frac{1}{t}\, \|f\|_1 ,$$

$$t \leq \fint_{I_j} f \leq 2t , \quad j = 1, 2, 3, \ldots$$

This allows us to represent f as a sum of two functions $f = g + b$ (good and bad) that are defined as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega, \\ \fint_{I_j} f & \text{if } x \in I_j \end{cases}$$

and

$$b(x) = \sum_{j=1}^{\infty} b_j(x) ,$$

where

$$b_j(x) = \left(f(x) - \fint_{I_j} f \right) \chi_{I_j}(x) .$$

Note that $g \in L^\infty \cap L^2$. Indeed, $0 \leq g \leq 2t$ and

$$\int_{\mathbb{R}} g(x)^2\, dx \leq 2t \int_{\mathbb{R}} g(x)\, dx = 2t\|f\|_1 , \tag{8.1}$$

Hence also $b \in L^2(\mathbb{R})$. Actually it is easy to see that

$$\int_{\mathbb{R}} |b_j(x)|^2\, dx \leq 4 \int_{I_j} |f(x)|^2\, dx ,$$

so the series $\sum_j b_j$ converges to b in L^2. Since $Hf = Hg + Hb$ we have

$$|\{|Hf| > t\}| \leq |\{|Hg| > t/2\}| + |\{|Hb| > t/2\}| .$$

The estimate for the first term on the right hand side is easy, which explains the name "good" for g. We have

$$\begin{aligned} |\{|Hg| > t/2\}| &\leq \left(\frac{2}{t}\right)^2 \int_{\mathbb{R}} |Hg(x)|^2 \, dx \\ &= \left(\frac{2}{t}\right)^2 \int_{\mathbb{R}} |g(x)|^2 \, dx \\ &\leq \frac{8}{t} \, \|f\|_1 \, . \end{aligned}$$

We used here the fact that H is an isometry on L^2 and inequality (8.1). The estimate for the second term is more involved (so the name "bad" for b). Let $2I_j$ be the interval concentric with I_j of twice the length. Define

$$\Omega^* = \bigcup_j 2I_j \, .$$

Clearly

$$|\Omega^*| \leq 2|\Omega| \leq \frac{2}{t} \, \|f\|_1 \, .$$

We have

$$\begin{aligned} |\{x \in \mathbb{R} : \, |Hb(x)| > t/2\}| &\leq |\Omega^*| + |\{x \notin \Omega^* : \, |Hb(x)| > t/2\}| \\ &\leq \frac{2}{t} \, \|f\|_1 + \frac{2}{t} \int_{\mathbb{R}\setminus\Omega^*} |Hb(x)| \, dx \, . \end{aligned}$$

Thus to complete the proof it remains to show that

$$\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)| \, dx \leq C\|f\|_1 \, .$$

Observe that

$$|Hb(x)| \leq \sum_{j=1}^{\infty} |Hb_j(x)| \quad \text{a.e.} \tag{8.2}$$

Indeed, for every k we have

$$\left| H\left(\sum_{j=1}^{k} b_j\right)\right| \leq \sum_{j=1}^{k} |Hb_j| \leq \sum_{j=1}^{\infty} |Hb_j| \quad \text{a.e.}$$

and it remains to use the fact that $\sum_{j=1}^{k} b_j$ converges to b in L^2 as $k \to \infty$ and that H is bounded in L^2. Inequality (8.2) gives

$$\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)| \leq \sum_j \int_{\mathbb{R}\setminus\Omega^*} |Hb_j(x)| \, dx \leq \sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)| \, dx \, .$$

Note that for $x \notin 2I_j$

$$Hb_j(x) = \frac{1}{\pi} \int_{I_j} \frac{b_j(y)}{x-y} \, dy \, ,$$

because b_j vanishes outside I_j, x is away from I_j and hence there is no singularity in the denominator. Let c_j be the center of the interval I_j. Since the integral of b_j equals 0 we have

$$\begin{aligned}\int_{\mathbb{R}\setminus 2I_j}|Hb_j(x)|\,dx &= \int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j}\frac{b_j(y)}{x-y}\,dy\right|\,dx\\ &= \int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j}b_j(y)\left(\frac{1}{x-y}-\frac{1}{x-c_j}\right)\,dy\right|\,dx\\ &\leq \int_{I_j}|b_j(y)|\left(\int_{\mathbb{R}\setminus 2I_j}\frac{|y-c_j|}{|x-y|\,|x-c_j|}\,dx\right)\,dy\\ &\leq \int_{I_j}|b_j(y)|\left(\int_{\mathbb{R}\setminus 2I_j}\frac{|I_j|}{|x-c_j|^2}\,dx\right)\,dy\,.\end{aligned}$$

The last inequality follows from a simple geometric observation that $|y-c_j|<|I_j|/2$ and $|x-y|>|x-c_j|/2$. Since

$$\int_{\mathbb{R}\setminus 2I_j}\frac{|I_j|}{|x-c_j|^2}\,dx=2$$

we have

$$\begin{aligned}\int_{\mathbb{R}\setminus\Omega^*}|Hb(x)|\,dx &\leq \sum_j\int_{\mathbb{R}\setminus 2I_j}|Hb_j(x)|\,dx\\ &\leq 2\sum_j\int_{I_j}|b_j(y)|\,dy\\ &\leq 4\|f\|_1\,.\end{aligned}$$

The proof is complete. □

In order to prove pointwise and L^p convergence of $H^\varepsilon f$ to Hf for $f\in L^p$, $1<p<\infty$ we needed to prove boundedness of the maximal Hilbert transform in L^p. We complement this result by showing that the maximal Hilbert transform is of weak type $(1,1)$.

Theorem 8.3. *H^* is of weak type $(1,1)$. More precisely*

$$|\{x:\mathbb{R}:\ H^*f(x)>t\}|\leq\frac{C}{t}\,\|f\|_1\quad\textit{for all } f\in L^1(\mathbb{R}).$$

Proof. The proof follows similar steps to those employed in the proof of Theorem 8.1. We can assume that $f\geq 0$. Fix $t>0$ and apply the Calderón-Zygmund decomposition to f and $\alpha=t$. As in the proof of Theorem 8.1 we decompose $f=g+b$, so

$$|\{H^*f>t\}|\leq|\{H^*g>t/2\}|+|\{H^*b>t/2\}|\,.$$

Note that $g \in L^1 \cap L^\infty$ and hence $g \in L^2$. Since $0 \le g \le 2t$ we have

$$\int_{\mathbb{R}} g(x)^2\,dx \le 2t \int_{\mathbb{R}} g(x)\,dx = 2t\|f\|_1\,.$$

Since H^* is bounded in L^2 (Theorem 5.16) we get

$$|\{H^*g > t/2\}| \le \frac{C}{t}\,\|f\|_1$$

by the same argument as in the proof of Theorem 8.1 and in order to estimate $|\{H^*b > t/2\}|$ it suffices to show that

$$|\{x \notin \Omega^* :\, H^*b(x) > t/2\}| \le \frac{C}{t}\,\|f\|_1\,.$$

Fix $x \notin \Omega^*$, $\varepsilon > 0$ and a function b_j. Recall that the function vanishes outside I_j. Denote by c_j the center of I_j. Clearly $x \notin 2I_j$ and one of the following conditions is satisfied

(a) $(x-\varepsilon, x+\varepsilon) \cap I_j = I_j$;
(b) $(x-\varepsilon, x+\varepsilon) \cap I_j = \emptyset$;
(c) $x-\varepsilon \in I_j$ or $x+\varepsilon \in I_j$.

In the first case $H^\varepsilon b_j(x) = 0$ and in the second one

$$H^\varepsilon b_j(x) = \int_{I_j} \frac{b_j(y)}{x-y}\,dy = \int_{I_j} \left(\frac{1}{x-y} - \frac{1}{x-c_j}\right) b_j(y)\,dy\,.$$

In either the first or second case we have

$$|H^\varepsilon b_j(x)| \le \frac{|I_j|}{|x-c_j|^2} \int_{I_j} |b_j(y)|\,dy \le \frac{2|I_j|}{|x-c_j|^2} \int_{I_j} |f(y)|\,dy\,. \tag{8.3}$$

In the third case, since $x \notin 2I_j$, we have $I_j \subset (x-3\varepsilon, x+3\varepsilon)$ and $|x-y| > \varepsilon/3$ for all $y \in I_j$, so

$$|H^\varepsilon b_j(x)| \le \int_{I_j} \frac{|b_j(y)|}{|x-y|}\,dy \le \frac{3}{\varepsilon} \int_{x-3\varepsilon}^{x+3\varepsilon} |b_j(y)|\,dy\,. \tag{8.4}$$

Since $b = \sum_j b_j \in L^1$ and the function

$$y \mapsto \frac{1}{x-y}\,\chi_{\{y:\,|x-y|\ge\varepsilon\}}$$

is bounded we have

$$H^\varepsilon b(x) = \sum_j \int_{|x-y|\ge\varepsilon} \frac{b_j(y)}{x-y}\,dy = \sum_j H^\varepsilon b_j(x)$$

for *every* x, so

$$|H^\varepsilon b(x)| \le \sum_j |H^\varepsilon b_j(x)|$$

everywhere. Adding up the estimates (8.3) and (8.4) for all j we obtain

$$|H^\varepsilon b(x)| \leq \underbrace{\sum_j \frac{2|I_j|}{|x-c_j|^2}\int_{I_j}|f(y)|\,dy}_{h(x)} + \underbrace{\frac{3}{\varepsilon}\int_{x-3\varepsilon}^{x+3\varepsilon}|b(y)|\,dy}_{\leq 18\mathcal{M}b(x)}\,.$$

Since this estimate is valid for every x and every $\varepsilon > 0$ taking the supremum over ε we have

$$H^*b(x) \leq h(x) + 18\mathcal{M}b(x)\,.$$

Thus

$$\begin{aligned} |\{x \notin \Omega^* : H^*b(x) > t/2\}| &\leq |\{x \notin \Omega^* : h(x) > t/4\}| \\ &+ |\{x \in \mathbb{R} : \mathcal{M}b(x) > t/72\}| \\ &\leq \frac{4}{t}\int_{\mathbb{R}\setminus\Omega^*}|h(x)|\,dx + \frac{C}{t}\int_{\mathbb{R}}|b(x)|\,dx \\ &\leq \frac{C}{t}\|f\|_1\,, \end{aligned}$$

because

$$\int_{\mathbb{R}}|b(x)|\,dx \leq 2\|f\|_1$$

and

$$\int_{\mathbb{R}\setminus\Omega^*}|h(x)|\,dx \leq \sum_j \int_{I_j}|f(y)|\,dy \underbrace{\int_{\mathbb{R}\setminus 2I_j}\frac{2|I_j|}{|x-c_j|^2}\,dx}_{4} \leq 4\|f\|_1\,.$$

The proof is complete. □

Applying Theorem 5.18 we immediately obtain

Theorem 8.4. *For all $f \in L^1(\mathbb{R})$ the limit*

$$Hf(x) = \lim_{\varepsilon\to 0} H^\varepsilon f(x)$$

exists a.e. and

$$|\{x \in \mathbb{R} : Hf(x) > t\}| \leq \frac{C}{t}\,\|f\|_1\,.$$

The last inequality is a slight improvement of Theorem 8.1, because it is true for all $f \in L^1$ and not only for $f \in L^1 \cap L^2$. Note that in this theorem the Hilbert transform of $f \in L^1$ is defined as the pointwise limit and not through the density argument.

8.2. **Calderón-Zygmund theory of singular integrals.** The method of the proof of boundedness of the Hilbert transform in L^p presented above easily generalizes to the case of multi-dimensional singular integrals.

DEFINITION. Let $K \in \mathcal{S}'_n$ be a tempered distribution such that

(a) $\hat{K} \in L^\infty(\mathbb{R}^n)$;
(b) K coincides with a locally integrable function in $\mathbb{R}^n \setminus \{0\}$;
(c) $K(x)$, $x \neq 0$ satisfies the *Hörmander condition*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq B$$

for some constant $B > 0$ and all $y \in \mathbb{R}^n$.

Then the convolution operator

$$T\varphi(x) = (K * \varphi)(x), \quad \varphi \in \mathcal{S}_n$$

is called a *singular integral*.

At this moment it is not clear how the class of singular integrals defined here is related to that defined in Section 7 as convolution with W_Ω. The two classes are not the same, but strongly related. We will investigate this relationship later, but now we will prove the main result about boundedness in L^p of the singular integrals defined *here*.

Observe that the condition (a) implies

$$\|T\varphi\|_2 \leq \|\hat{K}\|_\infty \, \|\varphi\|_2, \quad \varphi \in \mathcal{S}_n$$

so the operator T uniquely extends to a bounded operator in $L^2(\mathbb{R}^n)$.

Theorem 8.5 (Calderón-Zygmund)**.** *If T is a singular integral as defined above, then*

$$\|Tf\|_p \leq C_p \|f\|_p \quad \textit{for } f \in L^2 \cap L^p,\ 1 < p < \infty \tag{8.5}$$

and

$$|\{x \in \mathbb{R}^n : |Tf(x)| > t\}| \leq \frac{C}{t}\, \|f\|_1 \quad \textit{for } f \in L^1 \cap L^2. \tag{8.6}$$

Remark. In this theorem Tf for $f \in L^p \cap L^2$ is understood as the extension of T from $\mathcal{S}_n$ to L^2.

Proof. The proof is similar to that of Theorem 8.1 and 8.2. We will prove first (8.6). We can assume that $f \geq 0$. Fix $t > 0$ and apply the Calderón-Zygmund decomposition to f and $\alpha = t$. We have

$$f(x) \leq t \quad \text{for a.e. } x \notin \Omega = \textstyle\bigcup_j Q_j$$

$$|\Omega| \leq \frac{1}{t}\, \|f\|_1\,,$$

$$t \le \fint_{Q_j} f \le 2^n t, \quad j = 1, 2, 3, \ldots$$

Next we decompose $f = g + b$, where

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega, \\ \fint_{Q_j} f & \text{if } x \in Q_j \end{cases}$$

and

$$b(x) = \sum_j b_j(x)\,, \qquad b_j(x) = \left(f(x) - \fint_{Q_j} f \right) \chi_{Q_j}(x)\,.$$

Observe that $g \in L^1 \cap L^\infty$, $0 \le g(x) \le 2^n t$ and hence

$$\int_{\mathbb{R}^n} g(x)^2\, dx \le 2^n t \int_{\mathbb{R}^n} g(x)\, dx = 2^n t \|f\|_1\,.$$

Moreover

$$\int_{\mathbb{R}^n} |b_j|^2 \le 4 \int_{Q_j} |f|^2\,,$$

so the series $\sum_j b_j$ converges to b in L^2. Since T is bounded in L^2 we easily conclude that

$$|Tb(x)| \le \sum_j |Tb_j(x)| \quad \text{a.e.}$$

We have

$$|\{|Tf| > t\}| \le |\{|Tg| > t/2\}| + |\{|Tb| > t/2\}|\,.$$

The estimate for the first term on the right hand side is easy

$$|\{|Tg| > t/2\}| \le \left(\frac{2}{t}\right)^2 \int_{\mathbb{R}^n} |Tg|^2 \le \frac{C}{t^2} \int_{\mathbb{R}^n} |g|^2 \le \frac{C'}{t} \|f\|_1\,.$$

Let $Q_j^* = 2\sqrt{n} Q_j$ be a cube concentric with Q_j hose sides are $2\sqrt{n}$ times longer and let

$$\Omega^* = \bigcup_j Q_j^*\,.$$

Clearly

$$|\Omega^*| \le C|\Omega| \le \frac{C'}{t} \|f\|_1\,.$$

We have

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tb(x)| > t/2\}| &\le |\Omega^*| + |\{x \notin \Omega^* : |Tb(x)| > t/2\}| \\ &\le \frac{C}{t} \|f\|_1 + \frac{2}{t} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)|\, dx \\ &\le \frac{C}{t} \|f\|_1 + \frac{2}{t} \sum_j \int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\, dx \end{aligned}$$

and it remains to show that

$$\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\, dx \le C \int_{Q_j} |f(x)|\, dx\,.$$

Denote the common center of the cubes Q_j and Q_j^* by c_j. If $x \notin Q_j^*$, then

$$Tb_j(x) = \int_{Q_j} K(x-y) b_j(y)\, dx\,.$$

Since $\int_{Q_j} b_j(y)\, dy = 0$ we have

$$Tb_j(x) = \int_{Q_j} \big(K(x-y) - K(x-c_j)\big)\, b_j(y)\, dy$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\, dx &\le \int_{Q_j} |b_j(y)| \left(\int_{\mathbb{R}^n \setminus Q_j^*} |K(x-y) - K(x-c_j)|\, dx \right) dy \\ &\le B \int_{Q_j} |b_j(y)|\, dy \\ &\le 2B \int_{Q_j} |f(y)|\, dy\,,\end{aligned}$$

because an easy geometric investigation shows that

$$\mathbb{R}^n \setminus Q_j^* \subset \{x \in \mathbb{R}^n :\, |x - c_j| \ge 2|y - c_j|\}$$

and hence the above estimate follows from Hörmander's condition. This completes the proof of (8.6). Since the operator is bounded in L^2 and of weak type $(1,1)$, the Marcinkiewicz interpolation theorem implies (8.5) for $1 < p < 2$ and then the case $2 < p < \infty$ follows from the case $1 < p < 2$ by a duality argument, Theorem 4.6. $\square$

The conditions (a) and (c) in the definition of the singular integral seem difficult to verify, so we will investigate now sufficient and easy to verify conditions that imply (a) or (c). We will also compare the class of singular integrals considered in this section to that considered in Section 7. Let us start with the condition (c).

Proposition 8.6. *If $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is such that*

$$|\nabla K(x)| \le \frac{C}{|x|^{n+1}} \qquad x \ne 0 \tag{8.7}$$

then the function K satisfies the Hörmander condition.

Proof. Points on the interval connecting x to $(x-y)$ are of the form $x - ty$, $t \in [0,1]$, so for $|x| > 2|y|$

$$|x - ty| > \frac{|x|}{2}$$

and hence the mean value theorem gives

$$|K(x-y)-K(x)| \leq C\frac{|y|}{|x|^{n+1}}\,.$$

Thus the Hörmander condition follows upon integration in polar coordinates. □

The condition (8.7) is very easy to check and it covers majority of singular integrals that appear in applications. For example if $\Omega \in C^1(S^{n-1})$,

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$$

and

$$K(x) = \text{p.v.}\,\frac{\Omega(x/|x|)}{|x|^n}$$

then $\hat{K} \in L^\infty$ by Corollary 7.5 and

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \quad x \neq 0$$

since ∇K is homogeneous of degree $-(n+1)$. Thus

$$T_\Omega\varphi(x) = (K*\varphi)(x) = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n}\,\varphi(x-y)\,dy$$

is the singular integral in the sense described above.

Actually a weaker assumption about Ω implies the Hörmander condition.

Theorem 8.7. *Let $\Omega \in C(S^{n-1})$ be such that*

$$\int_0^1 \frac{\omega_\infty(t)}{t}\,dt < \infty\,, \tag{8.8}$$

where

$$\omega_\infty(t) = \sup\{|\Omega(\theta_1)-\Omega(\theta_2)| :\, |\theta_1-\theta_2| \leq t,\ \theta_1,\theta_2 \in S^{n-1}\}\,.$$

Then the function

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n}\,, \quad x \neq 0$$

satisfies the Hörmander condition. In particular the Hörmander condition is satisfied if Ω is Hölder continuous with exponent $0<\alpha\leq 1$.

Remark. (8.8) is called a Dini-type condition.

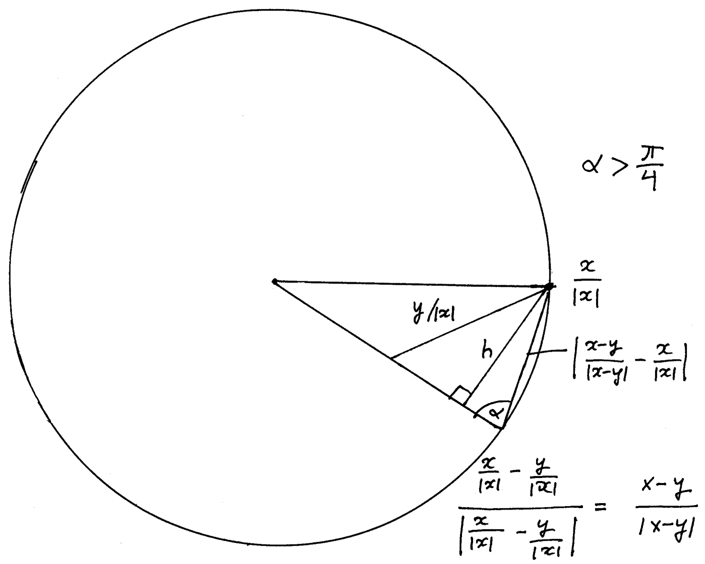
Proof. With the notation $\xi' = \xi/|\xi|$ we have

$$\begin{aligned}|K(x-y)-K(x)| &= \left|\frac{\Omega((x-y)')}{|x-y|^n} - \frac{\Omega(x')}{|x|^n}\right| \\ &\leq \frac{|\Omega((x-y)')-\Omega(x')|}{|x-y|^n} + |\Omega(x')|\left|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right| \qquad (8.9)\end{aligned}$$

The function Ω is bounded and the function $1/|x|^n$ satisfies the Hörmander condition by Proposition 8.6 so the integral of the second term in (8.9) over the region $\{|x| > 2|y|\}$ is bounded by a constant independent of y. To estimate the first term in (8.9) observe that

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| \le 2\, \frac{|y|}{|x|}\,.$$

The inequality easily follows from the picture



$$\frac{|y|}{|x|} \ge h = \sin\alpha \left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| \ge \frac{\sqrt{2}}{2} \left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right|,$$

Hence

$$\begin{aligned} \int_{|x|>2|y|} \frac{|\Omega((x-y)') - \Omega(x')|}{|x-y|^n}\, dx &\le \int_{|x|>2|y|} \frac{\omega_\infty(2|y|/|x|)}{(|x|/2)^n}\, dx \\ &= 2^n |S^{n-1}| \int_{2|y|}^{\infty} t^{n-1} \frac{\omega_\infty(2|y|/t)}{t^n}\, dt \\ &= 2^n n\omega_n \int_0^1 \frac{\omega_\infty(t)}{t}\, dt < \infty\,. \end{aligned}$$

The proof is complete. □

Thus if $\Omega \in C(S^1)$ satisfies

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$$

and the Dini-type condition (8.8), then

$$T_\Omega \varphi(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n}\, \varphi(x-y)\, dy$$

is the singular integral as defined above. In particular it is of strong type (p,p), $1<p<\infty$ and of weak type $(1,1)$. Observe that in Section 7 we proved strong type (p,p), $1<p<\infty$ of the singular integral associated with odd Ω that is just integrable, Theorem 7.6, and now we require some additional Dini-type regularity. It is natural to inquire if in the setting of Theorem 7.6 we can also prove the weak type $(1,1)$. Surprisingly, the answer is not known.

While we have seen situations where the Hörmander condition was satisfied, we still need to see when the condition (a) holds true. Now we will investigate sufficient conditions for the convolution with the principal value of $K \in L^1_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ to be a singular integral. This will include investigation of conditions for $\text{p.v.}\, K \in \mathcal{S}'_n$ and $\widehat{\text{p.v.}\, K} \in L^\infty$. We will consider the following properties

$$\int_{r<|x|<2r} |K(x)|\, dx \leq C_1 \tag{8.10}$$

for some $C_1>0$ and all $r>0$.

$$\left| \int_{r<|x|<R} K(x)\, dx \right| \leq C_2 \tag{8.11}$$

for some $C_2>0$ and all $0<r<R$.

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} K(x)\, dx \quad \text{exists and is finite.} \tag{8.12}$$

$$\int_{|x|>2|y|} |K(x-y)-K(x)|\, dx \leq C_3 \tag{8.13}$$

for some $C_3>0$ and all $y\in\mathbb{R}^n$.

The last condition is nothing but the Hörmander condition. Observe that (8.10) is equivalent to

$$\int_{|x|<r} |x||K(x)|\, dx \leq C_4 r \tag{8.14}$$

for some $C_4>0$ and all $r>0$. Indeed,

$$\begin{aligned} \int_{|x|<r} |x||K(x)|\, dx &= \sum_{k=0}^\infty \int_{2^{-(k+1)}r\leq|x|<2^{-k}r} |x||K(x)|\, dx \\ &\leq \sum_{k=0}^\infty 2^{-k} r C_1 = 2C_1 r\,, \end{aligned}$$

so we can take $C_4 = 2C_1$. In the other direction

$$\int_{r<|x|<2r} |K(x)|\,dx \le \int_{|x|<2r} \frac{|x|}{r}\,|K(x)|\,dx \le \frac{C_4 \cdot 2r}{r} = 2C_4$$

and we can take $C_1 = 2C_4$.

For example if $K \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies

$$|K(x)| \le \frac{C}{|x|^n}\,, \qquad |\nabla K(x)| \le \frac{C}{|x|^{n+1}}$$

then the conditions (8.10) and (8.13) follow from the integration in polar coordinates and Proposition 8.6.

The conditions (8.11) and (8.12) are true if the integral of K on every sphere centered at the origin is zero and in general they constitute a weaker form of this cancellation property. Observe that (8.12) is a necessary condition for p.v. K to be a tempered distribution. Indeed, if we take $\varphi \in C_0^\infty(\mathbb{R}^n)$ that is constant equal to 1 on the unit ball $B(0,1)$ we have

$$\text{p.v.}\,K[\varphi] = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)\,dx + \int_{|x|\ge 1} K(x)\varphi(x)\,dx\,.$$

Since the second integral on the right hand side is well defined and finite, the limit (8.12) must exist.

Proposition 8.8. *Suppose $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies (8.10) and (8.12). Then*

$$\text{p.v.}\,K[\varphi] = \lim_{\varepsilon\to 0}\int_{|x|<\varepsilon} K(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{S}_n$$

defines a tempered distribution.

Proof. For $\varphi \in \mathcal{S}_n$ we have

$$\begin{aligned}\int_{|x|\ge 1} |K(x)\varphi(x)|\,dx &\le \||x|\varphi\|_\infty \sum_{k=0}^{\infty} 2^{-k}\int_{2^k\le|x|<2^{k+1}} |K(x)|\,dx \\ &\le 2C_1\||x|\varphi\|_\infty\,.\end{aligned}$$

Since $|\varphi(x)-\varphi(0)| \le |x|\|\nabla\varphi\|_\infty$

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)(\varphi(x)-\varphi(0))\,dx = \int_{|x|<1} K(x)(\varphi(x)-\varphi(0))\,dx\,,$$

because of the estimate (8.14) with $C_4 = 2C_1$. Denoting

$$L = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)\,dx$$

we have

$$\text{p.v.}\,K[\varphi] = L\varphi(0) + \int_{|x|<1} K(x)(\varphi(x)-\varphi(0))\,dx + \int_{|x|\ge 1} K(x)\varphi(x)\,dx$$

and hence

$$|\text{p.v.}\, K[\varphi]| \le |L| \|\varphi\|_\infty + 2C_1(\|\nabla\varphi\|_\infty + \||x|\varphi\|_\infty)\,.$$

The proof is complete. $\square$

For $K \in L^1_{\rm loc}(\mathbb{R}^n \setminus \{0\}$ and $0 < \varepsilon < R$ we define the truncated kernels by

$$K_{\varepsilon,R} = K\chi_{\{\varepsilon<|x|<R\}}$$

and

$$K_\varepsilon = K\chi_{\{|x|>\varepsilon\}}\,.$$

Theorem 8.9. *If $K \in L^1_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ satisfies (8.10), (8.11) and (8.13), then $\widehat{K_{\varepsilon,R}} \in L^\infty(\mathbb{R}^n)$ and*

$$\|\widehat{K_{\varepsilon,R}}\|_\infty \le C \tag{8.15}$$

for some constant $C > 0$ independent of ε and R. Moreover $K_\varepsilon \in \mathcal{S}'_n$, $\widehat{K_\varepsilon} \in L^\infty(\mathbb{R}^n)$ and

$$\|\widehat{K_\varepsilon}\|_\infty \le C$$

with the same constant as in (8.15).

Proof. Observe that for $|y| \le \varepsilon/2$ the truncated kernel $K_{\varepsilon,R}$ satisfies a version of the Hörmander condition in a form described below

$$\begin{aligned} K_{\varepsilon,R}(x-y) - K_{\varepsilon,R}(x) &= (K(x-y)-K(x))\chi_{\{\varepsilon<|x|<R\}}(x) \\ &+ K(x-y)\left(\chi_{\{\varepsilon<|x-y|<R\}}(x) - \chi_{\{\varepsilon<|x|<R\}}(x)\right)\,, \end{aligned}$$

so

$$\begin{aligned} \int_{\mathbb{R}^n} |K_{\varepsilon,R}(x-y) - K_{\varepsilon,R}(x)|\,dx &\le \int_{\varepsilon<|x|<R} |K(x-y)-K(x)|\,dx \\ &+ \int_{\mathbb{R}^n} |K(x)| \left|\chi_{\{\varepsilon<|x|<R\}}(x) - \chi_{\{\varepsilon<|x+y|<R\}}(x)\right|\,dx \\ &\le \int_{\varepsilon<|x|<R} |K(x-y)-K(x)|\,dx + \int_{\varepsilon-|y|<|x|<\varepsilon+|y|} |K(x)|\,dx \\ &+ \int_{R-|y|<|x|<R+|y|} |K(x)|\,dx \\ &\le C_3 + 4C_1\,. \end{aligned} \tag{8.16}$$

Indeed, a simple geometric consideration shows that the symmetric difference of the two annuli $\{\varepsilon < |x| < R\}$ and $\{\varepsilon < |x+y| < R\}$ is contained in

$$\{\varepsilon - |y| < |x| < \varepsilon + |y|\} \cup \{R - |y| < |x| < R + |y|\}\,.$$

and the last inequality follows from (8.10), (8.13) and the fact that $|y| \le \varepsilon/2$.

To estimate the Fourier transform of $K_{\varepsilon,R}$ observe first that

$$|\widehat{K_{\varepsilon,R}}(0)| = \left| \int_{\varepsilon<|x|<R} K(x)\, dx \right| \le C_2 \,,$$

so we may assume that $\xi \neq 0$. Assume for a moment that $\varepsilon < |\xi|^{-1} < R$ and denote $r = |\xi|^{-1}$. We have

$$\widehat{K_{\varepsilon,R}}(\xi) = \widehat{K_{\varepsilon,r}}(\xi) + \widehat{K_{r,R}}(\xi) \,.$$

First we will estimate the second term.

$$\begin{aligned}
\widehat{K_{r,R}}(\xi) &= \int_{\mathbb{R}^n} K_{r,R}(x) e^{-2\pi i x\cdot\xi}\, dx \\
&= \int_{\mathbb{R}^n} K_{r,R}(x-y) e^{-2\pi i (x-y)\cdot\xi}\, dx \\
&= e^{2\pi i y\cdot\xi} \int_{\mathbb{R}^n} K_{r,R}(x-y) e^{-2\pi i x\cdot\xi}\, dx \,.
\end{aligned}$$

Taking $y = \frac{1}{2}\xi|\xi|^{-2}$ we have $\exp(2\pi i y\cdot\xi) = -1$ and hence

$$\begin{aligned}
|\widehat{K_{r,R}}(\xi)| &= \left| \frac{1}{2} \int_{\mathbb{R}^n} \big(K_{r,R}(x) - K_{r,R}(x-y)\big)\, e^{-2\pi i x\cdot\xi}\, dx \right| \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \big|K_{r,R}(x-y) - K_{r,R}(x)\big|\, dx \\
&\le \frac{1}{2} C_3 + 2C_1 \,,
\end{aligned}$$

by (8.16) since $|y| = r/2$.

For the other term we have

$$\begin{aligned}
|\widehat{K_{\varepsilon,r}}(\xi)| &\le \left| \int_{\varepsilon<|x|<r} K(x) \left(e^{-2\pi i x\cdot\xi} - 1\right)\, dx \right| + \left| \int_{\varepsilon<|x|<r} K(x)\, dx \right| \\
&\le 2\pi|\xi| \int_{|x|<r} |K(x)|\, |x|\, dx + C_2 \\
&\le 2\pi|\xi| \cdot 2C_1 r + C_2 \\
&= 4\pi C_1 + C_2 \,.
\end{aligned}$$

In the case $|\xi|^{-1} \le \varepsilon$ or $|\xi|^{-1} \ge R$ we directly estimate $\widehat{K_{\varepsilon,R}}(\xi)$ without splitting it into two parts. If $|\xi|^{-1} \le \varepsilon$ the estimate goes as that for $\widehat{K_{r,R}}$ and if $|\xi|^{-1} \ge R$ as that for $\widehat{K_{\varepsilon,r}}$. We leave easy details to the reader.

Now it is time to take care of K_ε. Observe first that $K_\varepsilon \in \mathcal{S}_n'$ by an estimate similar to that at the beginning of the proof of Proposition 8.8. We have

$$\widehat{K_\varepsilon}[\varphi] = K_\varepsilon[\hat{\varphi}] = \lim_{R\to\infty} K_{\varepsilon,R}[\hat{\varphi}] = \lim_{R\to\infty} \widehat{K_{\varepsilon,R}}[\varphi]$$

and hence

$$|\widehat{K_\varepsilon}[\varphi]| \leq C\|\varphi\|_1$$

for all $\varphi \in \mathcal{S}_n'$. Thus $\varphi \mapsto \widehat{K_\varepsilon}[\varphi]$ extends to a bounded functional on $L^1(\mathbb{R}^n)$, so $\widehat{K_\varepsilon} \in L^\infty$ and $\|\widehat{K_\varepsilon}\|_\infty \leq C$. The proof is complete. □

Theorem 8.10 (Calderón-Zygmund)**.** *Suppose that* $K \in L^1_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ *satisfies (8.10), (8.11), (8.12) and (8.13). Then* p.v. K *is a tempered distribution and the convolution with* p.v. K*,*

$$T\varphi = (\mathrm{p.v.}\, K) * \varphi, \quad \varphi \in \mathcal{S}_n$$

is a singular integral as defined in this section. In particular

$$\|Tf\|_p \leq C_p\|f\|_p, \quad \textit{for } f \in L^p \cap L^2,\ 1 < p < \infty,$$

$$|\{x \in \mathbb{R}^n :\, |Tf(x)| > t\}| \leq \frac{C}{t}\, \|f\|_1, \quad \textit{for } f \in L^1 \cap L^2.$$

Proof. We already proved in Proposition 8.8 that p.v. $K \in \mathcal{S}_n'$ and according to Theorem 8.5 it remains to show that $\widehat{\mathrm{p.v.}\, K} \in L^\infty$. For $\varphi \in \mathcal{S}_n$ we have

$$\left|\widehat{\mathrm{p.v.}\, K}[\varphi]\right| = \left|\lim_{\varepsilon\to 0} K_\varepsilon[\hat{\varphi}]\right| = \left|\lim_{\varepsilon\to 0} \widehat{K_\varepsilon}[\varphi]\right| \leq C\|\varphi\|_1\,,$$

so $\widehat{\mathrm{p.v.}\, K} \in L^\infty$ with $\|\widehat{\mathrm{p.v.}\, K}\|_\infty \leq C$. □

Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA, hajlasz@pitt.edu